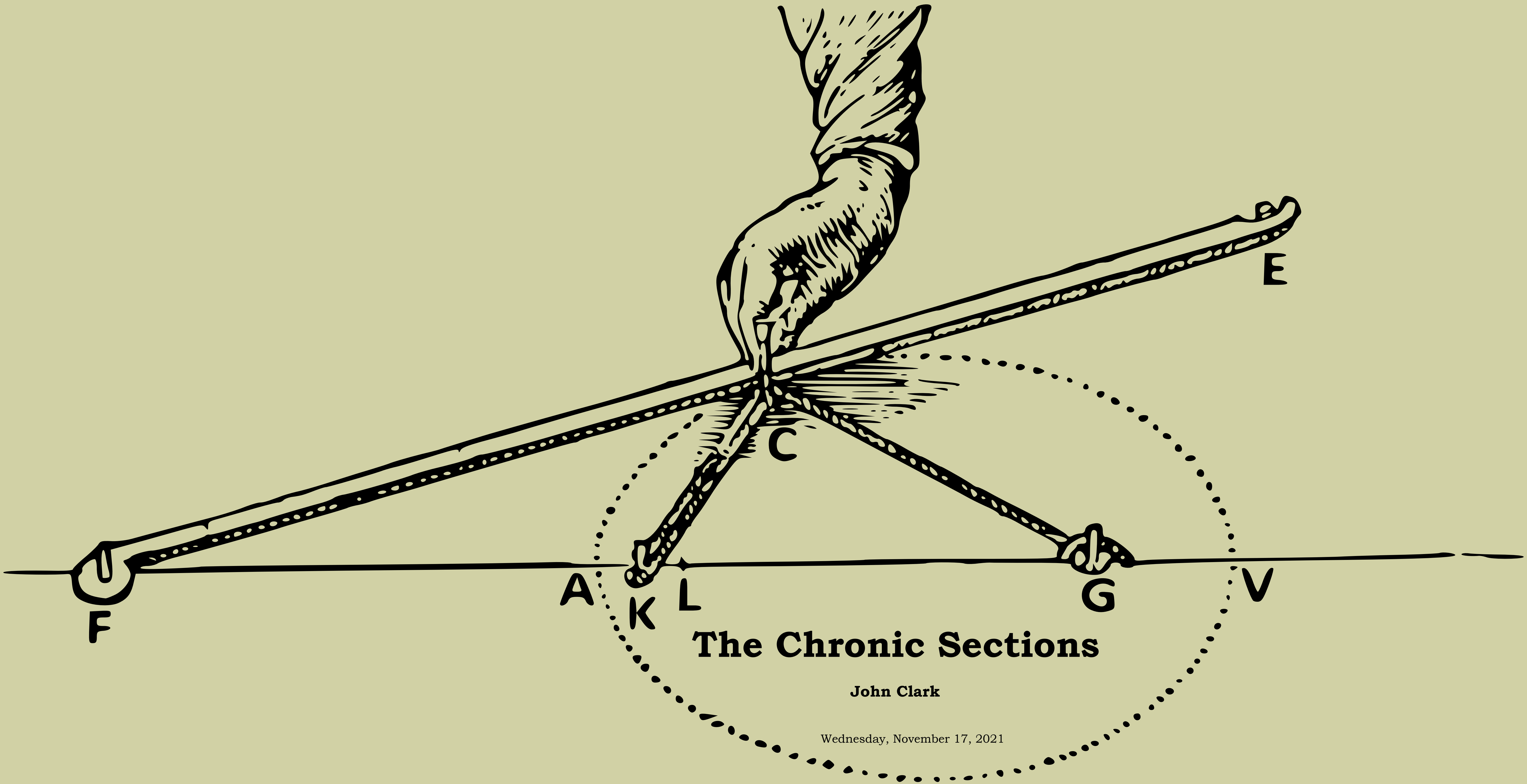


The Chronic Sections

John Clark

Wednesday, November 17, 2021

The Chronic Sections

Thursday, November 18, 2021

What is the important concept in regard to the so-called conic sections? In case one did not notice it, the *Elements* of Euclid deals with what may be called static figures, the simple figure. Conic Sections embodied the minds step into the world of dynamic figures, figures produced by the elementary figures themselves. What this means is that the division in geometry is not from plane figures to solids, but from static figures to dynamic figures. How one can claim that we divide between plane and solid figures while working on the same piece of paper is rather odd. Every binary grammar, and every grammar is binary, can be written on the same piece of paper. Only an idiot goes looking for a n^{th} dimensional notebook in order to write the story of *Alice in Wonderland*. An engineer looks for ways to produce material things, a grammarian sticks with their plain old flat piece of paper and writing tools.

Binary recursion affords us four distinct systems of grammar: Common Grammar, Arithmetic, Algebra and Geometry. The correct learning of grammar recognizes that we are learning this binary grammar matrix. So let me go over once again, in a concise format, how we go from the definition of a thing, any thing as a binary construct which produces four systems of grammar and how, by a simple revelation, we come to understand that binary recursion produces a binary result, no matter how it is transformed into these four grammar systems, the result being, we always say the same things about the same things. We are not those who pretend to be geniuses who believe that they can invent languages, grammars, which all contradict one another like it is normal to wake up in the morning and dry our hair with our microwave oven.

Therefore, if we are to become literate, we take a figure, a simple dynamic figure which, draws a locus. We then write the equations to that figure. After we have the equations, we write the geometrical figure which processes algebra to draw the very same loci. I call it Jacob's Ladder, since I discovered it, and since when laid out it does look like a ladder: (Don't make me take my toys home!) Thus, we have the figure produce the algebra, then we use the algebra to produce the figure.

This process will allow us, after a life-time of searching, to answer the question, which came first, the chicken or the egg, and simply answer, in an identity, there is no first or second, an equality does not produce a difference.

As I said long, long ago, a literate child should be able to prove that the Einstein's and Hawkins, of the world, who claim to be able to predict even the birth and death of the Universe, of reality itself, yet who never once demonstrated that they could predict the movement of their own hand, were simply illiterate.

In order to step up from the static figure, defined in terms of two tools, straightedge and compass, to step up to dynamic geometry, we transform these two tools into the intelligible which allows us to follow the literate, that geometry, as a grammar, affords that grammar like any other, through binary recursion. The segment is our unit, the compass allows us the recursion of that unit, so if one cannot see that the compass, recursion, loci, are all synonymous as we only have, by binary fact, two concepts to master, then they cannot even imagine a solution to the *Delian Problem*, while, on the other hand, I solved it not only using a static figure, but show how to do exponential progressions for any root series. Our job in learning how to become literate, is simply how to learn to say the same things, about the same things, no matter which of our four systems of grammar we use, our aim is to pair them in order to derive the greatest utility from binary recursion in order to do our biologically defined job. At some point

in our mental evolution, we finally have come to the conclusion, that the recursion of the compass, is still a product of that very same compass.

After all that hard thinking, I need a vodka and coke and while I relax, I will go over it again. As the compass allows us to recursively apply the unit, then is it not also inferred that we are also admitting the result of recursively, dynamically, applying that very same compass? Maybe I need a beer chaser to help carry on. Moving on:

And all this education has an added bonus, it will give you a start on how to help cure all those mental invalids, scribbling with so called logical operators, symbolic algebra, and algebraic geometry, Cartesian Geometry and Trigonometry, etc. Someone has to help them grow a brain and that someone could be you!

Meanwhile, I will continue, at least for a while, to develop this project until I am somewhat satisfied with it.

Demonstration ordering (subject to change without notice)

01. The Conchoid

02. The Conic

03. Curve of the Equation

04. Infinity

05. The Perfect Heart

06 Rose Bud

07 Tear Drop

08 The X-Files

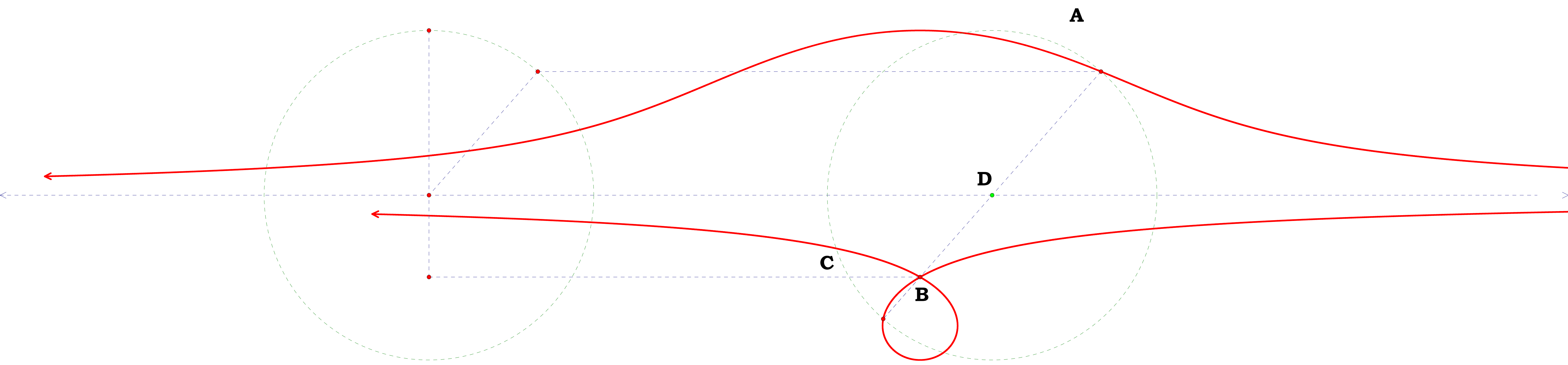
The Conchoid

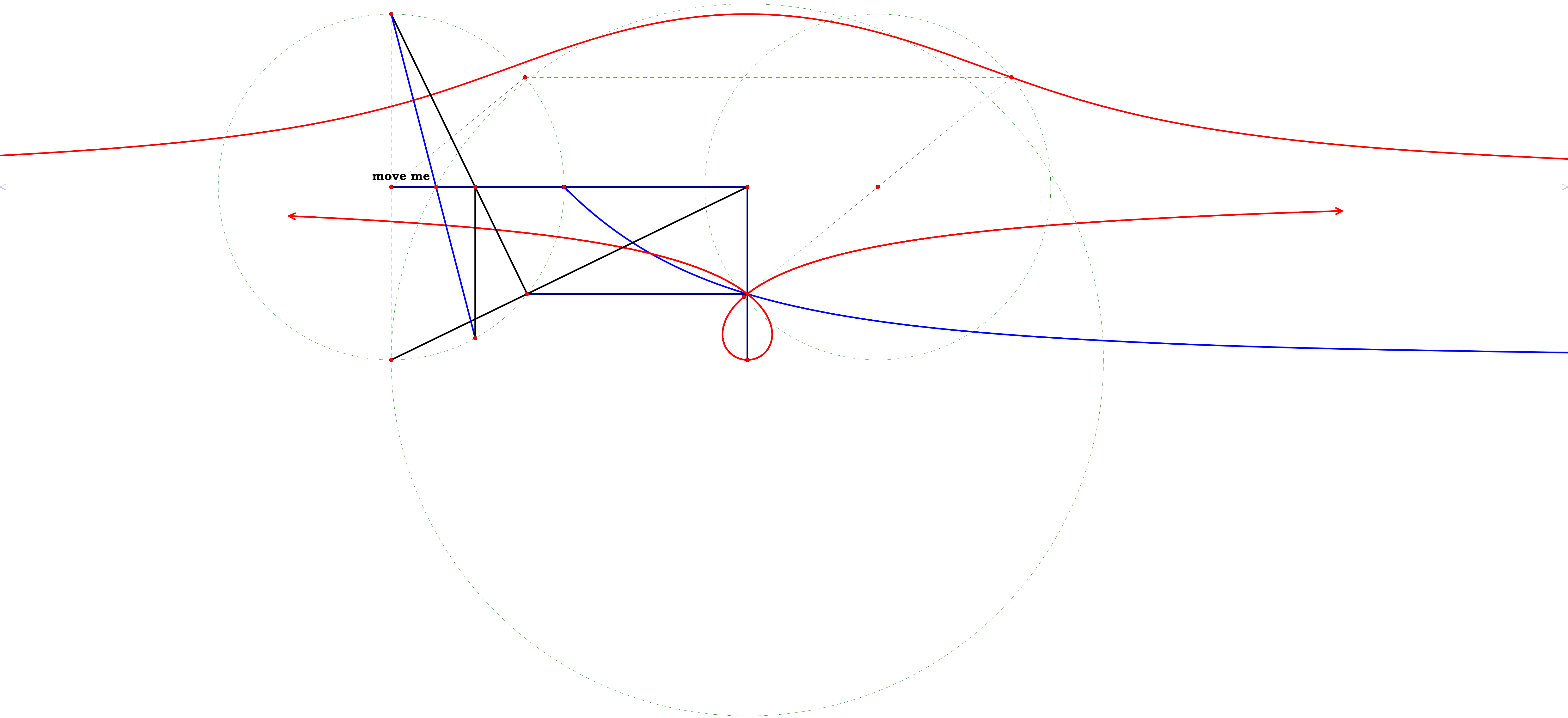
Thursday, November 18, 2021

Sir Thomas Heath can give you any real world information on the conchoid, I just decided that it should be illustrated in a clear manner and with a bit more of a literate explanation. I may eventually write up the equations for it, but not today.

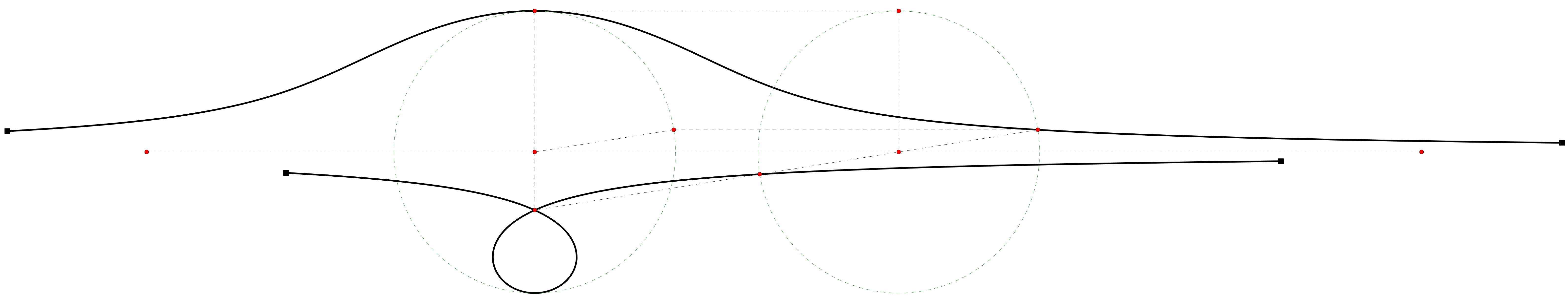
As one can see from the illustration, it can be written up as a simple ratio, but since I have absolutely no use for it, it is highly unmotivating.

The Conchoid.

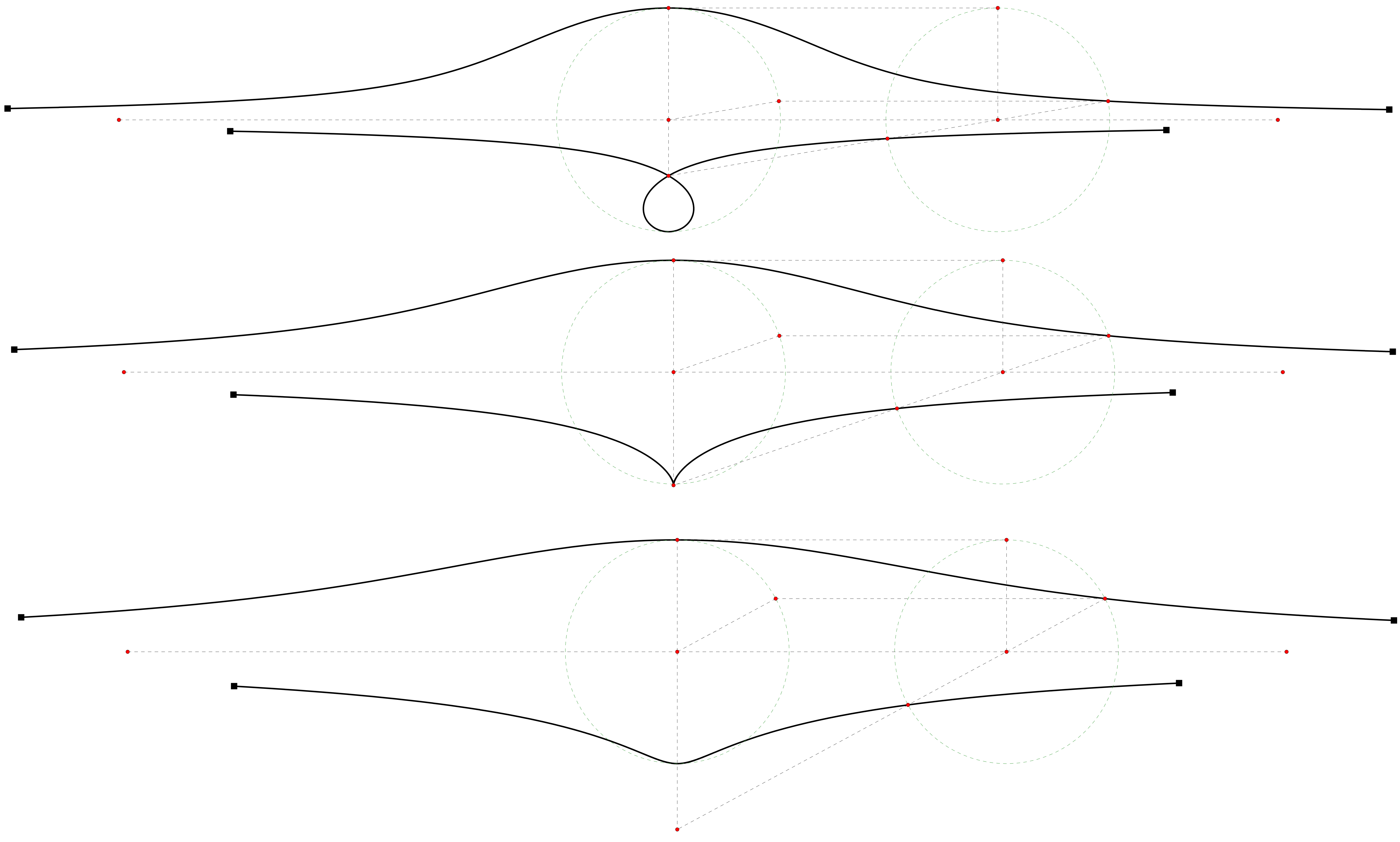



move me

The Conchoid



Changing the pole length.



The Conchoid

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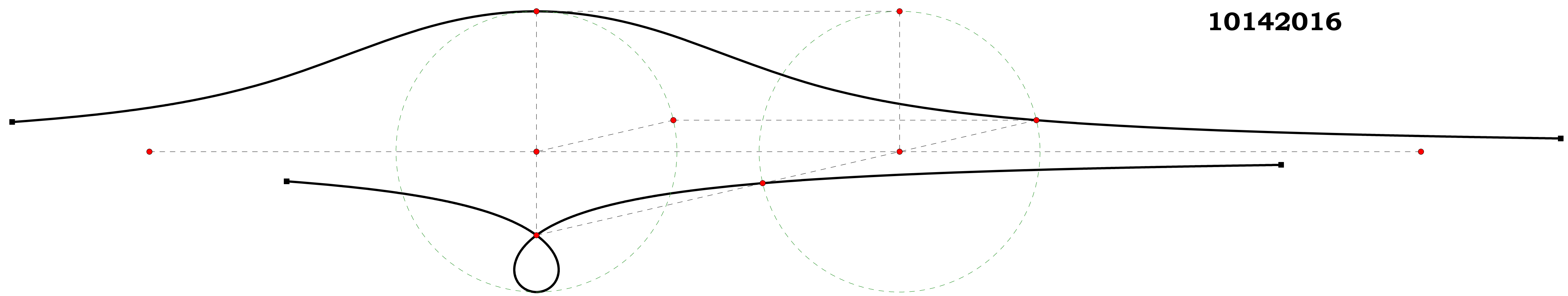
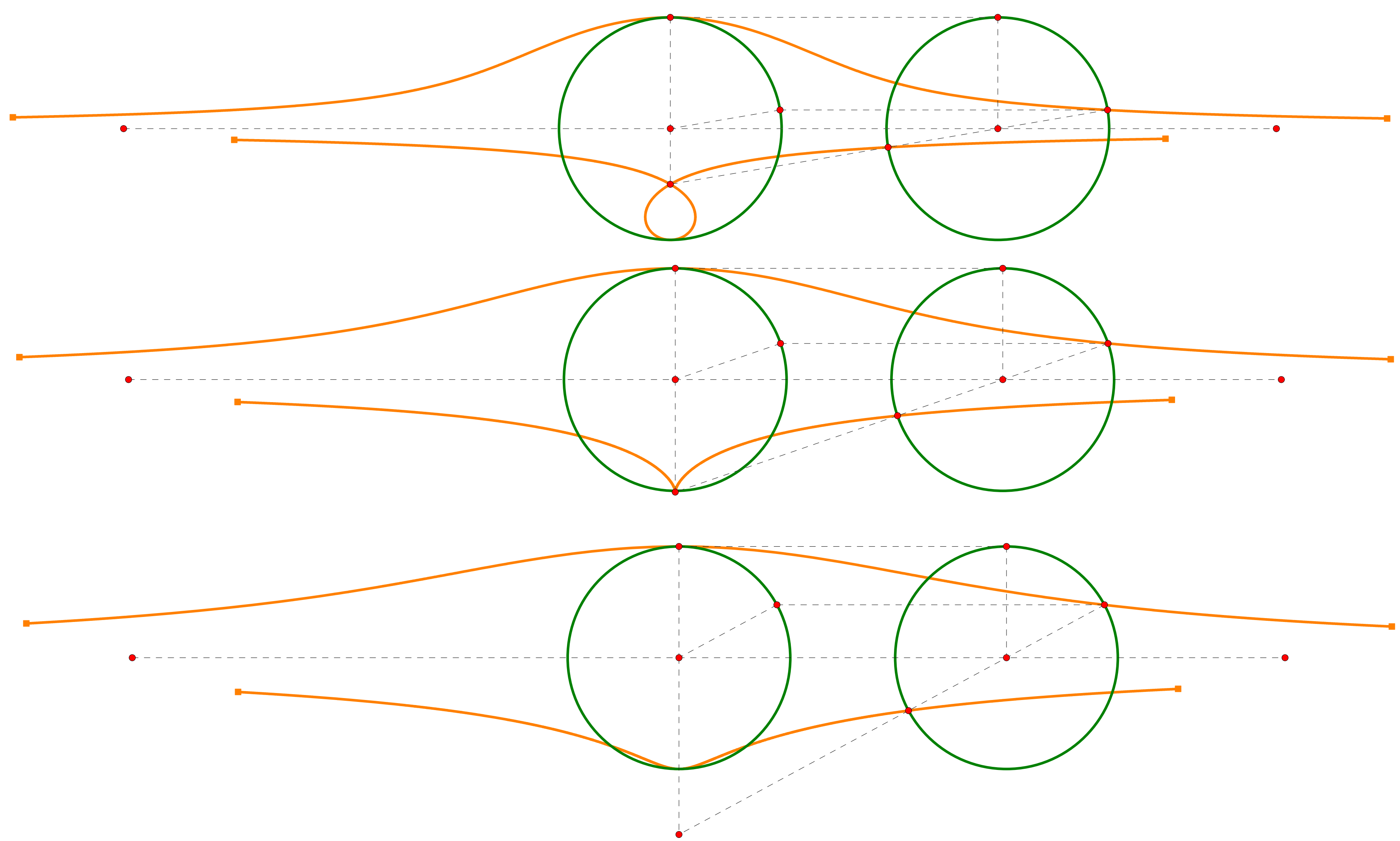


I found some of my old notes on Heath. Heath mentions this in his Euclid's *Elements*. His graphic and explaination was not very good, so I decided to draw what he was describing and then maybe I will, at some time, tackle writing up the equations and doing some research on how it was used. See what I can come up with. I do not understand why it is thought that it produces different curves when every one of them all have the same equation. If one wants to think they can classify all the curves by how they look, what will they do with BAM? Way too much appearance over reality. It makes it easier to understand when one can gather it all in with a glance.

Seems to me, if one can draw it using the language, one can write the equations. If one can write the equations, one can write the equations to what it was used for.



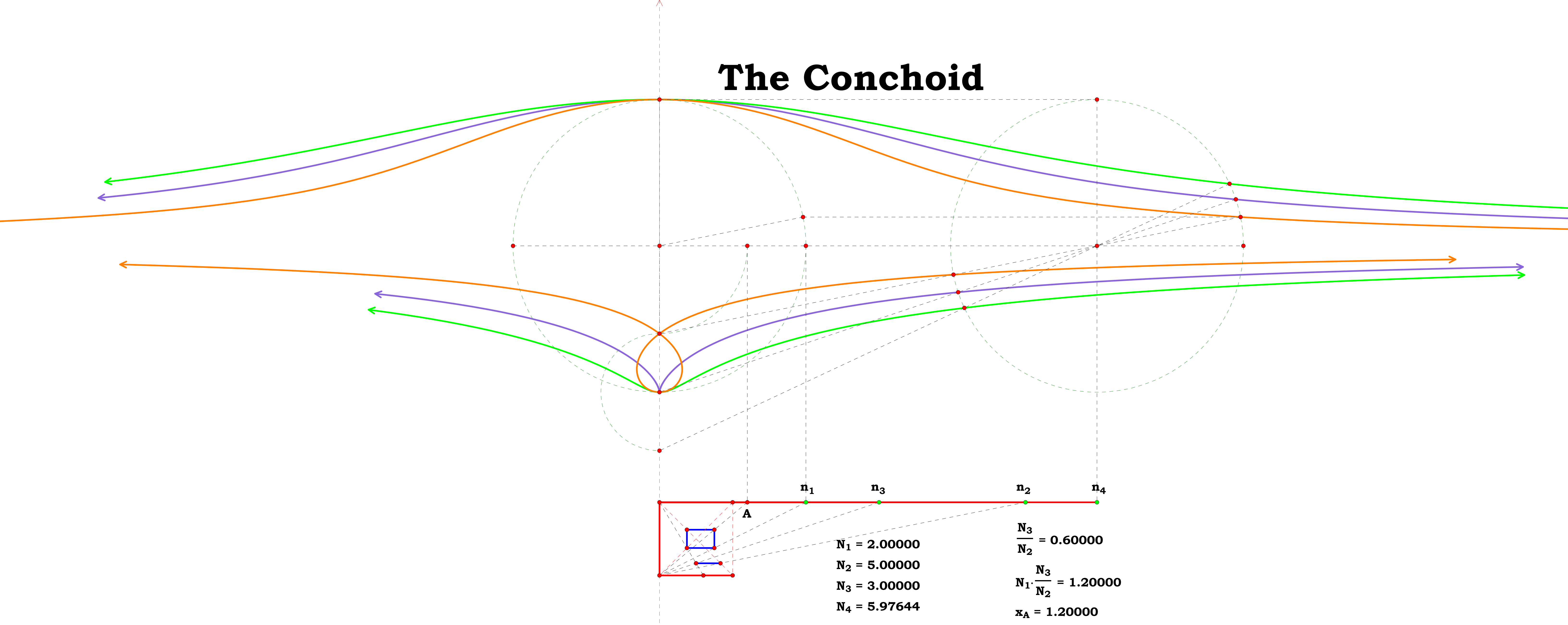
As the figure uses the unit for construction, it is a valid geometric construction just like the circle. When drawn correctly, it is easy to find any point one desires.

If one can actually solve all the problems one wants with this, as they say, then I guess that is that, all one has to do is fill in the blanks.

So, this should be a nice little diversion if I get the opportunity to write it up.

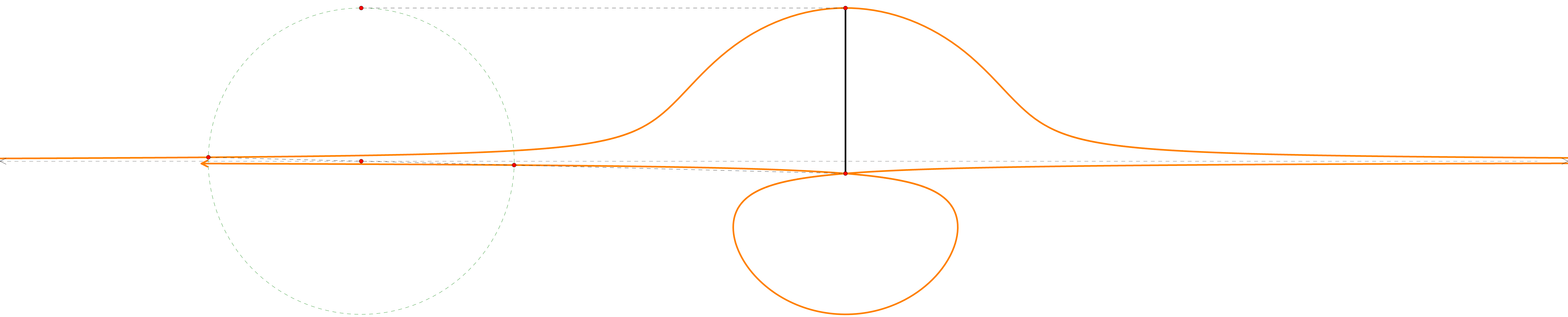
It is not any more difficult to write up than the conic sections.

The Conchoid



The Oxford dic. has a better def. than what I seen online. Not exactly sure that *a curve* cuts it though; it is actually two distinct curves. I do not believe that if one groups two curves, they then can call it a curve of two branches. Sounds rather illiterate to me. Secondly, the line from the pole does not determine the division between the curves and asymtote unless one is a banker, a lawyer or a government official cutting things in half. That pole line is through two different places of the curves. And quite frankly, it is not quartic curve, it is just another ratio. There are several vantage points from which to write an equation. All of the equations can be referenced from the starting circle's line and the distance from the centers. Only an ill trained bumbling mathematician would dice something simple up and call it quartic, why not myoptic?

noun Mathematics *a plane quartic curve* consisting of two separate branches either side of and asymptotic to a central straight line (the asymptote), such that if a line is drawn from a fixed point (the pole) to intersect both branches, the part of the line falling between the two branches is of constant length and is exactly bisected by the asymptote.



So, after a couple of thousand years, no one can still describe the thing right? A chonchoid is a group of two curves produced using a fixed point and the limits of the diameter of a circle which is recursively moved from that pole point along the horizontal axis. Trim it up a bit, and one has the making of Yosimity's Sams face looking for Bugs Bunny.

The Conic

Friday, November 19, 2021

I have been working on a project for conics, even produced a Compendium of Conics, and How to Draw a Conic.

I found, when I went to study the literature, the poverty of the authors. They could not describe it correctly, nor correctly explain its equation, much of the same is on the internet. The amount of obfuscation involving one of the simplest curves is rather enlightening, but how it is, that after two-thousand four hundred years, the correct expression, in a simple and clear fashion, never evolved is rather disturbing.

I do, however, find solace in being able to use the situation to justify my coffee addiction.

The conic sections resolve to the a given ratio, and like all ratios expressible between 0 and 1 and whose sum is 1, and its recursion in relation to a right triangle.

Unit. $AB := 1$

Given.

$z := .6666666$

$x := 1.60090$

Conic by Deduction

Historically, the conic is defined by its eccentricity called e. However, one can say that a conic is a locus formed by a right triangle using a point from the vertex of a point common to the hypotenuse and opposite side of that triange at 1 and the ratio set by any point z between 0 and 1. The term concic section is derived from a metaphor of one example of the curve and so called solids. It is also traditional to botch not give the equation of the conic, or to give it incorrectly.

Given z, describe the conic.

Descriptions.

$Bz := AB - z \quad e := \frac{Bz}{z} \quad BC := \frac{AB \cdot Bz}{z}$

$Dx := BC \cdot x \quad BE := Dx \quad By := BE$

$Bx := x - AB \quad xy := \sqrt{By^2 - Bx^2}$

Definitions.

$Bz - (1 - z) = 0 \quad e := \frac{1 - z}{z} \quad BC - \frac{1 - z}{z} = 0$

$LatusRectum := BC \quad LatusRectum - e = 0$

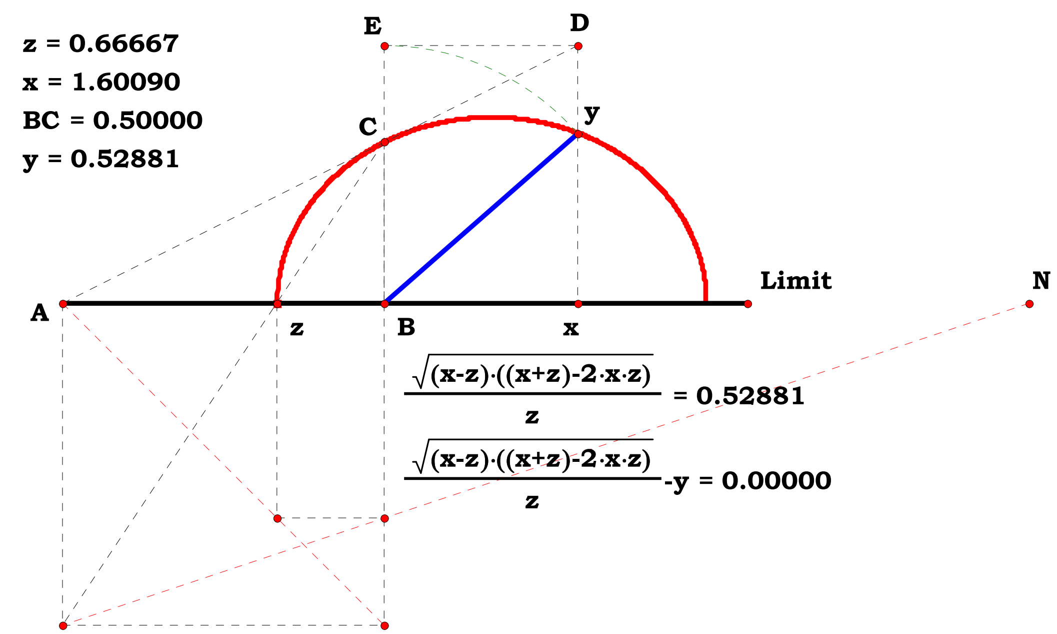
$Dx - \frac{x \cdot (1 - z)}{z} = 0 \quad BE - \frac{x \cdot (1 - z)}{z} = 0$

$By - \frac{x \cdot (1 - z)}{z} = 0 \quad Bx - (x - 1) = 0$

$xy - \frac{\sqrt{(x - z) \cdot (x + z - 2 \cdot x \cdot z)}}{z} = 0$

$y := xy$

$y - \frac{\sqrt{(x - z) \cdot (x + z - 2 \cdot x \cdot z)}}{z} = 0$



Unit. $AC := 1$

Given.

$D := 1.45843$

$x := 2$

Conic by Induction, i.e. x and D are givens.

Descriptions.

$$AB := 1 - \frac{1}{x} \qquad BC := AC - AB \qquad CG := \frac{AC \cdot BC}{AB}$$

$$DE := CG \cdot D \qquad CF := DE \qquad Cy := DE \qquad CD := D - 1$$

$$y := \sqrt{Cy^2 - CD^2}$$

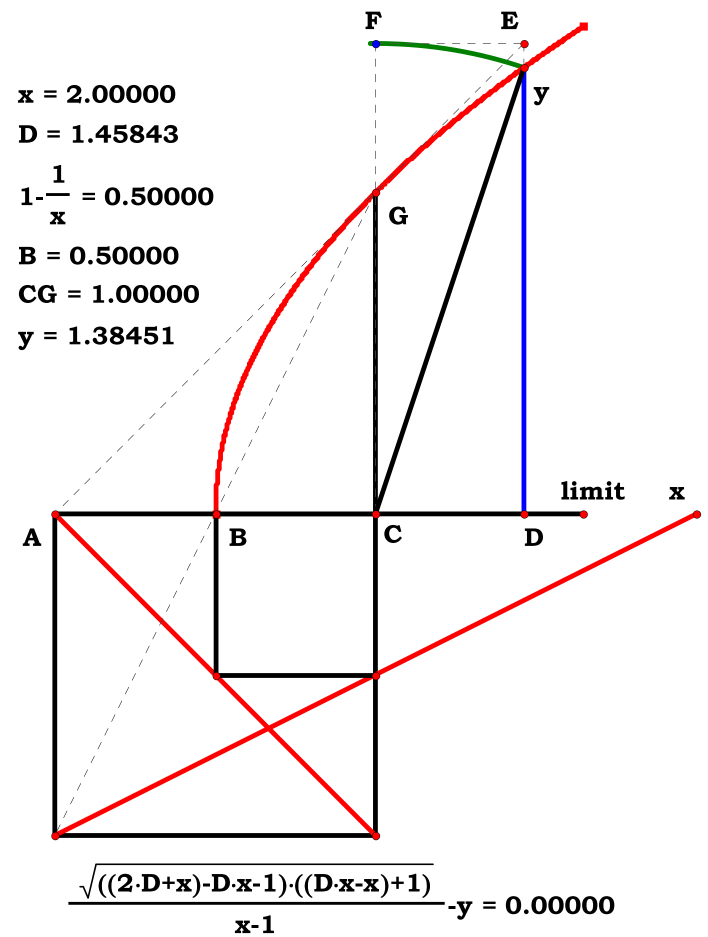
Definitions.

$$AB - \left(1 - \frac{1}{x}\right) = 0 \qquad BC - \frac{1}{x} = 0 \qquad CG - \frac{1}{x-1} = 0$$

$$DE - \frac{D}{x-1} = 0 \qquad CF - \frac{D}{x-1} = 0 \qquad Cy - \frac{D}{x-1} = 0$$

$$CD - (D - 1) = 0$$

$$y - \frac{\sqrt{(2 \cdot D + x - D \cdot x - 1) \cdot (D \cdot x - x + 1)}}{(x-1)} = 0$$



Computational Geometry

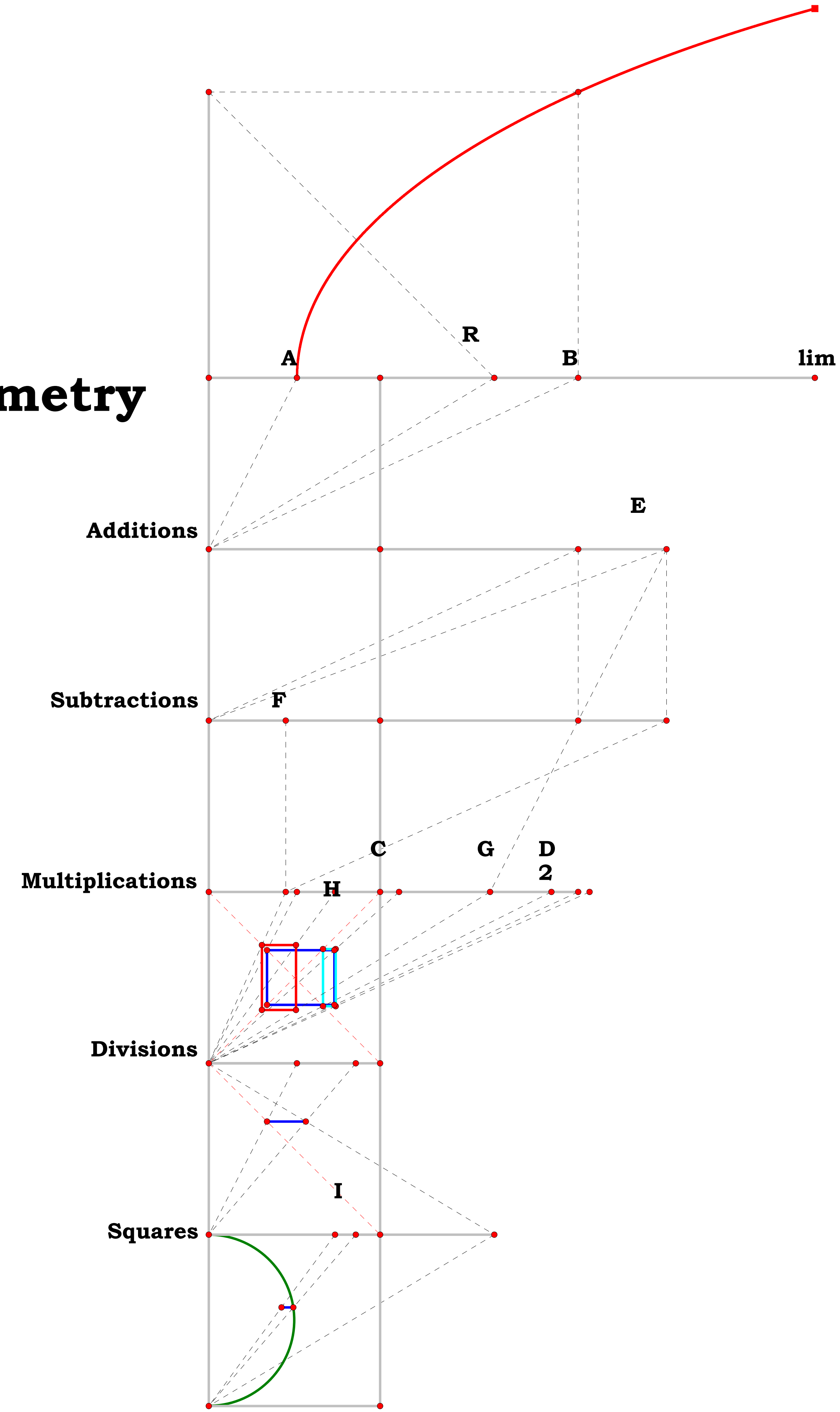
$$\frac{\sqrt{((B-A)\cdot((A+B)-2\cdot A\cdot B))}}{A} = 1.66742$$

R = 1.66742

$$\frac{\sqrt{((B-A)\cdot((A+B)-2\cdot A\cdot B))}}{A} - R = 0.00000$$

B = 2.15715
A = 0.51523
C = 1.11143
$A\cdot B - C = 0.00000$
D = 2.22286
$2\cdot A\cdot B - D = 0.00000$
E = 2.67238
$(A+B)-E = 0.00000$
F = 0.44952
$(A+B)-2\cdot A\cdot B = 0.44952$
G = 1.64191
B-A = 1.64191

$(B-A)\cdot((A+B)-2\cdot A\cdot B) = 0.73806$
H = 0.73806
$x_I = 0.85911$
$\sqrt{((B-A)\cdot((A+B)-2\cdot A\cdot B))} = 0.85911$



Wrapping a Can.

$R = 2.87422$

$$\frac{\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)}}{A}-R = 0.00000$$

$B = 0.46541$

$A = 0.02632$

$C = 0.01225$

$A\cdot B = 0.01225$

$D = 0.02450$

$2\cdot A\cdot B-D = 0.00000$

$E = 0.49173$

$(A+B)-E = 0.00000$

$F = 0.46723$

$(A+B)-2\cdot A\cdot B = 0.46723$

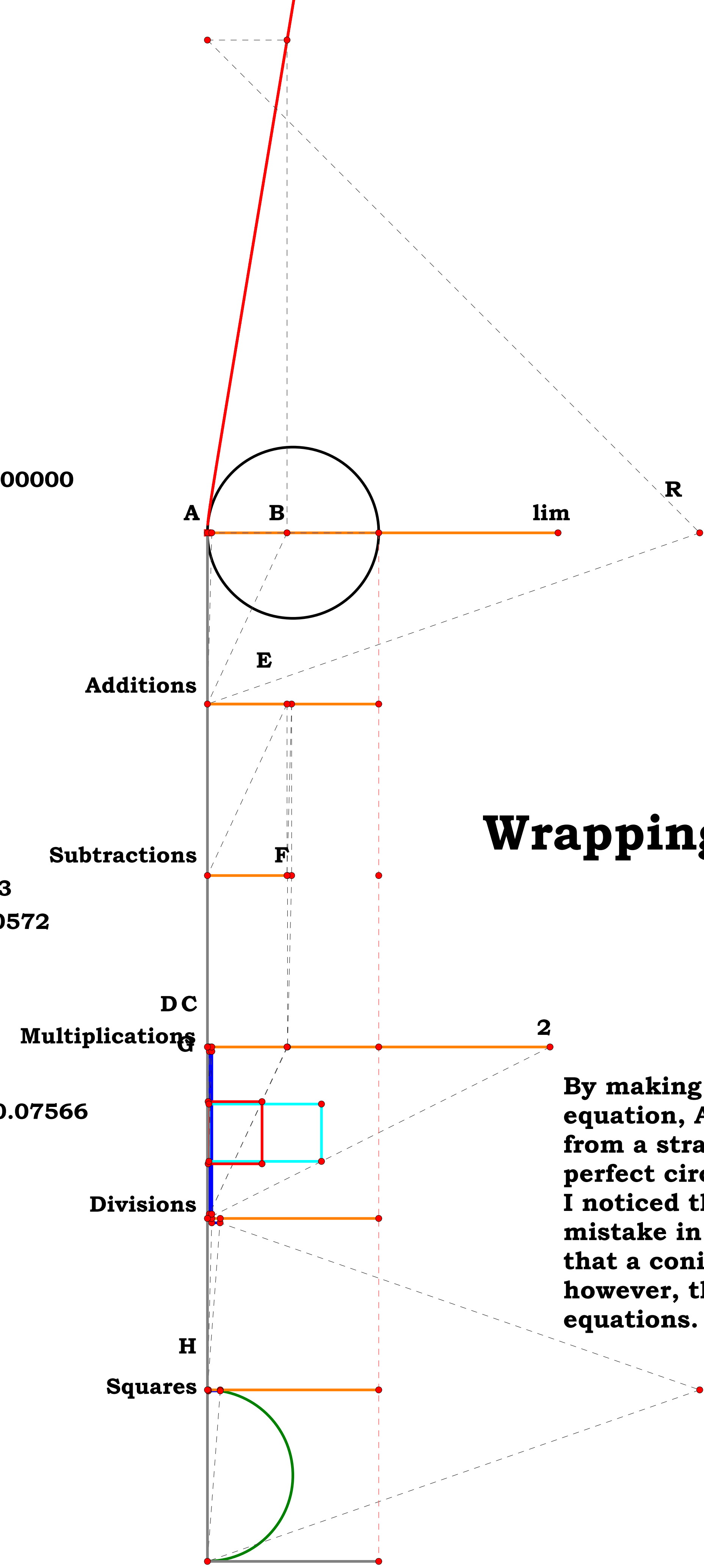
$((A+B)-2\cdot A\cdot B)\cdot C = 0.00572$

$G = 0.00572$

$\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)} = 0.07566$

$H = 0.07566$

$\frac{H}{A} = 2.87422$



By making a small change in our equation, A - B at the top to AxB, we start from a straight line, and end up with a perfect circle instead of a point.
I noticed that Discartes made the mistake in his Geometry of imagining that a conic ended up in a circle, however, these are two different equations.

$$R = 1.09202$$

$$\frac{\sqrt{(A \cdot B) \cdot ((A+B) - 2 \cdot A \cdot B)}}{A} - R = 0.00000$$

$B = 1.41475$

$A = 0.52939$

$C = 0.74895$

$A \cdot B = 0.74895$

$D = 1.49790$

$2 \cdot A \cdot B - D = 0.00000$

$E = 1.94414$

$(A+B) - E = 0.00000$

$F = 0.44623$

$(A+B) - 2 \cdot A \cdot B = 0.44623$

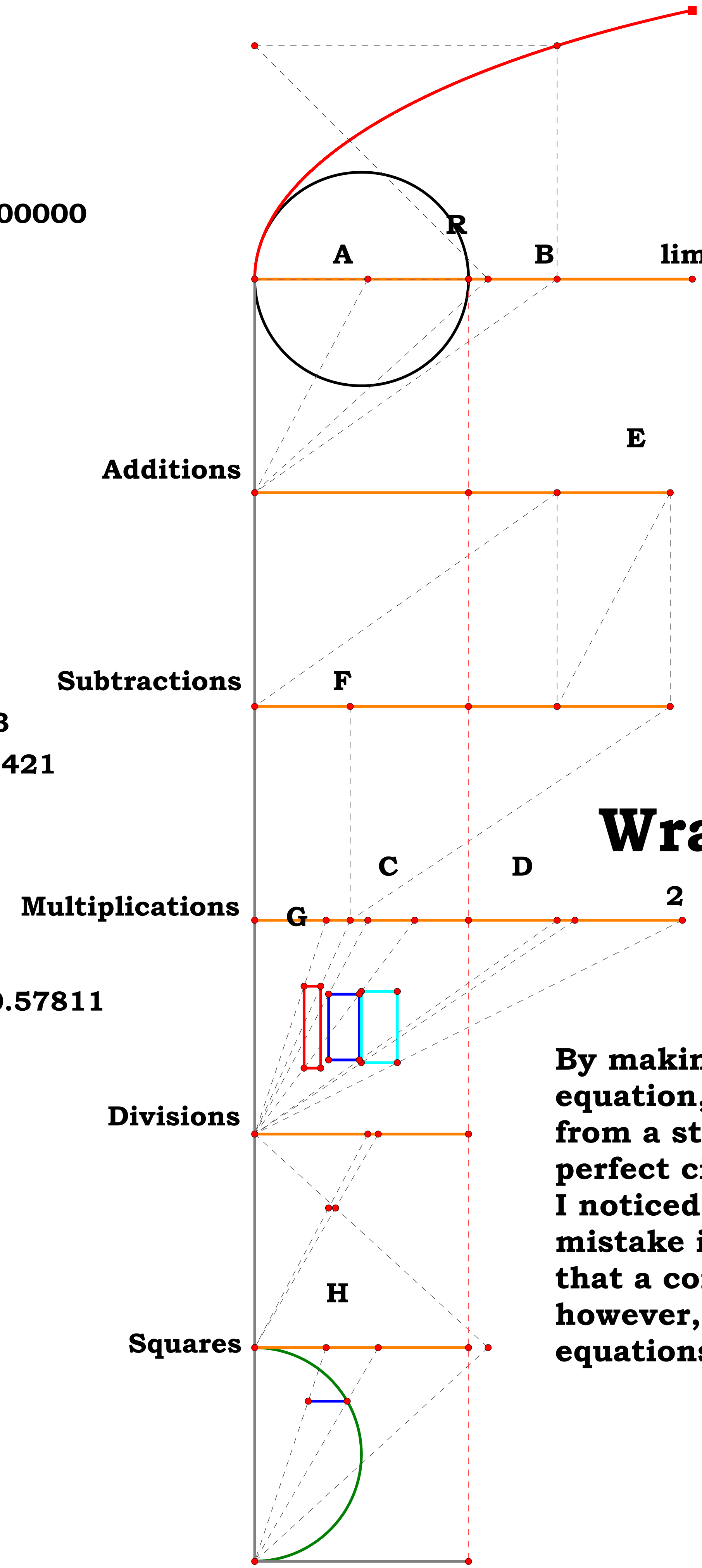
$((A+B) - 2 \cdot A \cdot B) \cdot C = 0.33421$

$G = 0.33421$

$\sqrt{(A \cdot B) \cdot ((A+B) - 2 \cdot A \cdot B)} = 0.57811$

$H = 0.57811$

$$\frac{H}{A} = 1.09202$$



Wrapping a Can.

By making a small change in our equation, A - B at the top to AxB, we start from a straight line, and end up with a perfect circle instead of a point.
I noticed that Discartes made the mistake in his Geometry of imagining that a conic ended up in a circle, however, these are two different equations.

$R = 0.55900$

$$\frac{\sqrt{(A \cdot B) \cdot ((A+B) - 2 \cdot A \cdot B)}}{A} - R = 0.00000$$

$B = 0.46541$

$A = 0.77281$

$C = 0.35967$

$A \cdot B = 0.35967$

$D = 0.71935$

$2 \cdot A \cdot B - D = 0.00000$

$E = 1.23822$

$(A+B) - E = 0.00000$

$F = 0.51887$

$(A+B) - 2 \cdot A \cdot B = 0.51887$

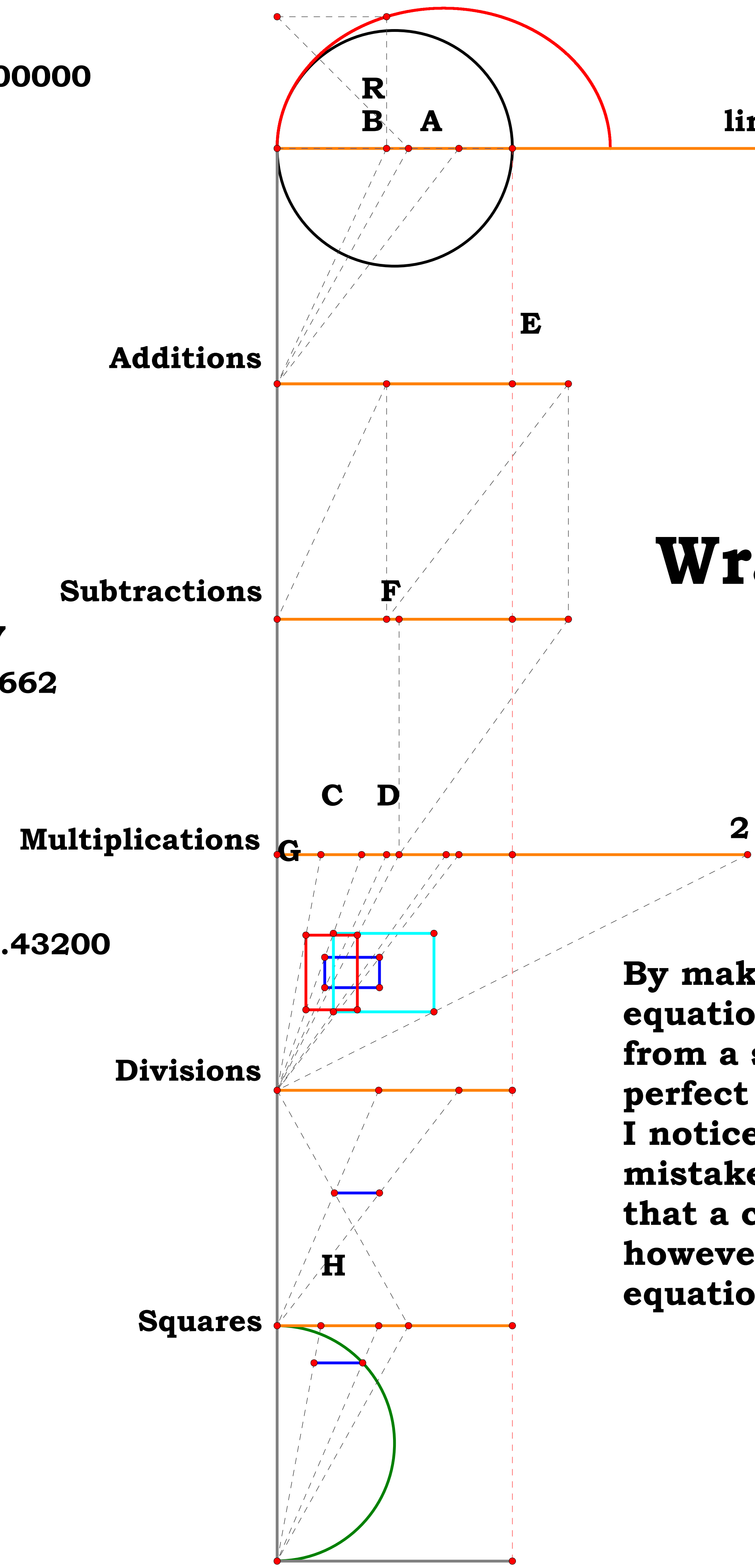
$((A+B) - 2 \cdot A \cdot B) \cdot C = 0.18662$

$G = 0.18662$

$\sqrt{(A \cdot B) \cdot ((A+B) - 2 \cdot A \cdot B)} = 0.43200$

$H = 0.43200$

$\frac{H}{A} = 0.55900$



Wrapping a Can.

By making a small change in our equation, A - B at the top to AxB, we start from a straight line, and end up with a perfect circle instead of a point.
I noticed that Discartes made the mistake in his Geometry of imagining that a conic ended up in a circle, however, these are two different equations.

$A = 1.00000$

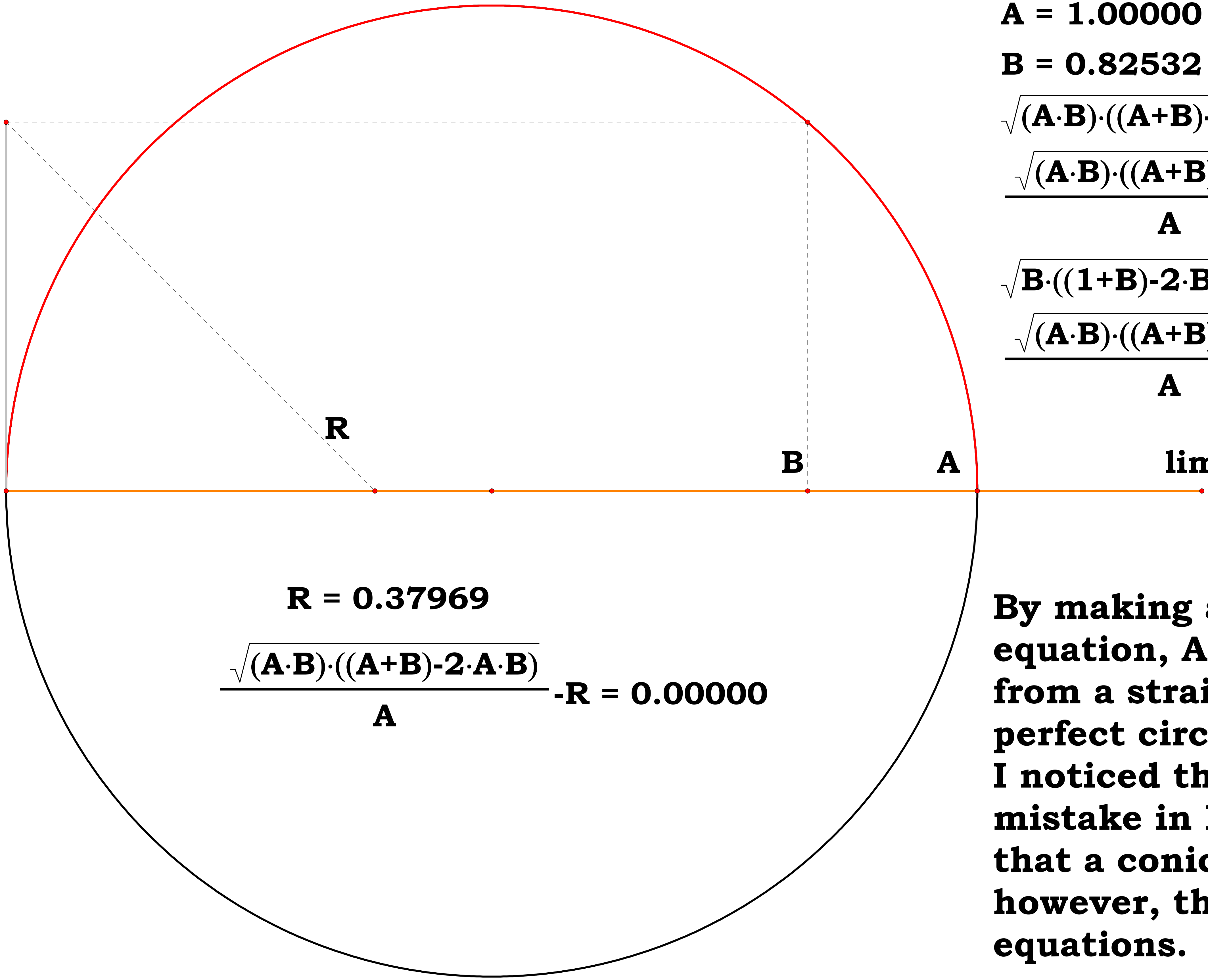
$B = 0.82532$

$$\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)} = 0.37969$$

$$\frac{\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)}}{A} = 0.37969$$

$$\sqrt{B\cdot((1+B)-2\cdot B)} = 0.37969$$

$$\frac{\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)}}{A}-\sqrt{B\cdot((1+B)-2\cdot B)} = 0.00000$$



$R = 0.37969$

$$\frac{\sqrt{(A\cdot B)\cdot((A+B)-2\cdot A\cdot B)}}{A}-R = 0.00000$$

By making a small change in our equation, A - B at the top to AxB, we start from a straight line, and end up with a perfect circle instead of a point.
I noticed that Discartes made the mistake in his Geometry of imagining that a conic ended up in a circle, however, these are two different equations.

Wrapping a Can.

The Curves of the Equation

Friday, November 19, 2021

Our most basic curve is, of course, the circle and it has a very simple equation, $\sqrt{A \times B} = C$ unless, of course, you are claiming to a Cartesian who actually thinks drawing graphing paper is the results of the concept of binary recursion. The sum of the two segments for the circle, just like in the conic, is simply 1. $A + B = 1$. Thus, the opening stipulation for conics is the unit and the unit ratio and the different ways to express it in drawing one's equations. If one think about it just a little, they will then realize that there is no actual special defining characteristic for a conic which is not applicable to all of geometry. The term Conic, is factually jargon, a result of a mind trying to grasp a distinction between static and dynamic geometry.

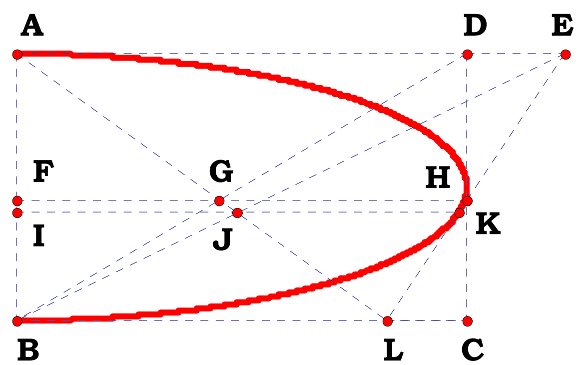
Conics, actually example, what happens when a mind gets stuck, it loops over and over, and never realizes how to break out and develop the idea it is stuck on. Conics represent both a solid pedantry on examining a particular figure, and an impoverished mind which cannot develop the ideas being presented to it.

One can actually, then, draw the circle using a straight line as I demonstrate in the Curve of the Equation. Imagine that, geometry which is the result of straight lines and their ratios, what a concept, I think I will patent it, copyright it, embellish it with gibberish, and make a fortune! And so, to alleviate all boredom, I use that circle to demonstrate a method of exponential series because while we mostly think of the equation for the circle as something which takes place in the interior of the circle, we have it take a trip outside the box, or igloo if you like.

Keep these figures in mind when you get to the X-Files, as one can then combine the figures to fully develop their machinations both inductively and deductively.

Curve of the Equation.

$$\frac{2\cdot N_3\cdot N_2 - 2\cdot N_2}{{N_3}^2 - 2\cdot N_3 + 2} = 4.2$$



$N_1 \equiv 9999999999$

$N_2 \equiv 7$

$N_3 \equiv 4$

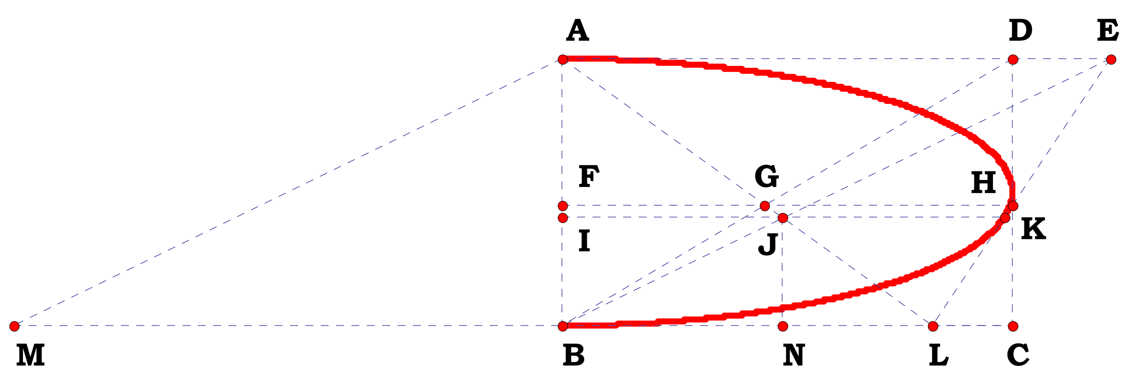
$AB := N_1$

$AD := N_2 \quad BC := AD$

$AF := \dfrac{AB}{N_3}$

$BF := AB - AF \quad FG := \dfrac{BC\cdot BF}{AB} \quad BL := \dfrac{FG\cdot AB}{AF} \quad CL := BC - BL$

$DE := \dfrac{CL\cdot AF}{BF} \quad AE := AD + DE \quad \sqrt{BL\cdot AE} - AD = 0 \quad JN := \dfrac{AB\cdot BL}{AE + BL}$



$BN := \dfrac{AE\cdot JN}{AB} \quad IK := 2\cdot BN \quad IK - \dfrac{2\cdot N_3\cdot N_2 - 2\cdot N_2}{{N_3}^2 - 2\cdot N_3 + 2} = 0 \quad IK = 4.2$

$00 = 2.34950$ cm

$N_00 = 10.11767$ cm

$N_10 = 7.59534$ cm

$N_20 = 13.47764$ cm

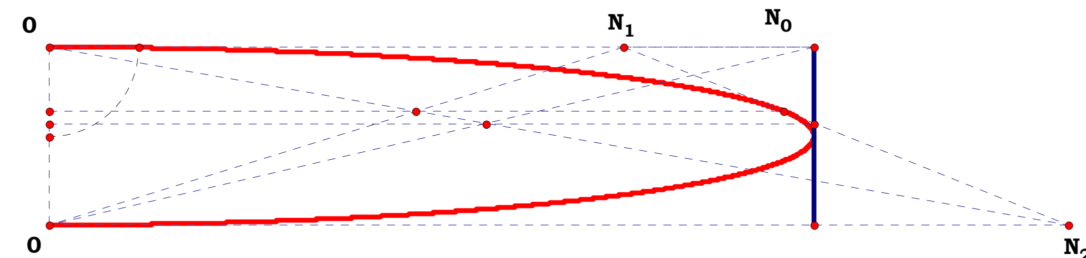
$N_10{\cdot}N_20{-}N_00^2 = 0.00000$ cm²

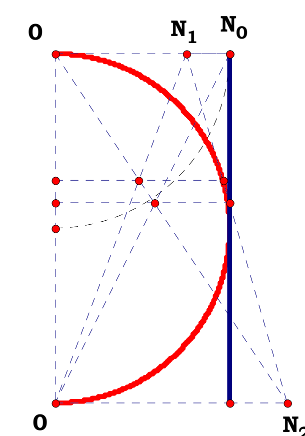
$N_0 = 4.30631$

$N_1 = 3.23275$

$N_2 = 5.73638$

$N_1{\cdot}N_2{-}N_0^2 = 0.00000$





$00 = 4.61433$ cm

$N_00 = 2.30717$ cm

$N_10 = 1.73199$ cm

$N_20 = 3.07335$ cm

$N_10{\cdot}N_20{-}N_00^2 = 0.00000$ cm²

$N_0 = 0.50000$

$N_1 = 0.37535$

$N_2 = 0.66604$

$N_1{\cdot}N_2{-}N_0^2 = 0.00000$

After one acquires the equation, even if one put 0 in for the unit of the geometry, the equation remains the same for a circle and ellipse. This means, that no matter how one envision the circle, larger than largest or not at all its definition remains the same.

Proportion is independent of the naming convention.

$$\frac{2 \cdot N_3 \cdot N_2 - 2 \cdot N_2}{{N_3}^2 - 2 \cdot N_3 + 2} = 4.2$$

00 = 2.34950 cm

$N_0$0 = 10.11767 cm

$N_1$0 = 7.59534 cm

$N_2$0 = 13.47764 cm

$N_1$0·$N_2$0-$N_0$0^2 = 0.00000 cm^2

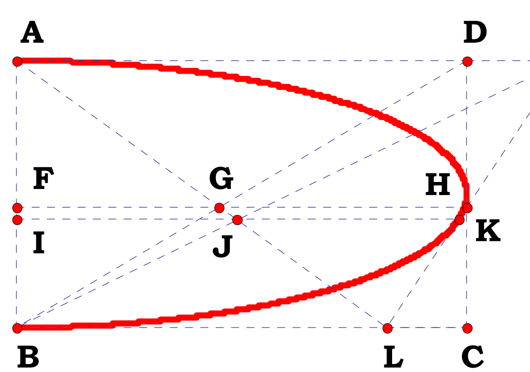
N_0 = 4.30631

N_1 = 3.23275

N_2 = 5.73638

$N_1 \cdot N_2 - N_0^2$ = 0.00000

Curve of the Equation.



$N_1 \equiv 0$

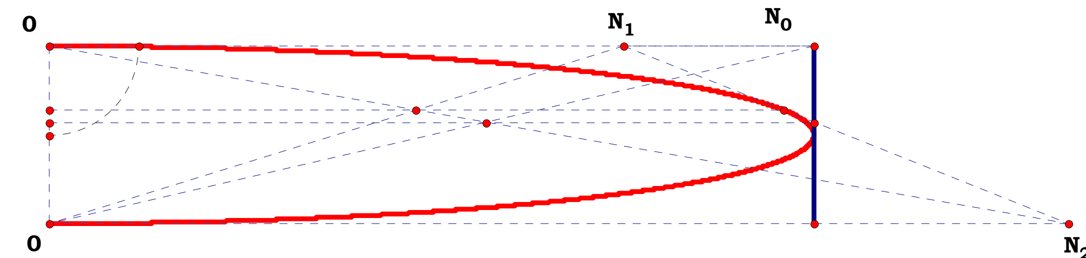
$N_2 \equiv 7$

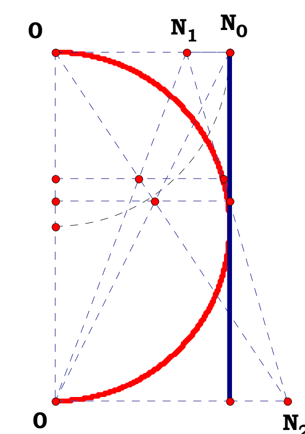
$N_3 \equiv 4$

$AB := N_1$

$AD := N_2 \quad BC := AD$

$AF := \dfrac{AB}{N_3}$





00 = 4.61433 cm

$N_0$0 = 2.30717 cm

$N_1$0 = 1.73199 cm

$N_2$0 = 3.07335 cm

$N_1$0·$N_2$0-$N_0$0^2 = 0.00000 cm^2

N_0 = 0.50000

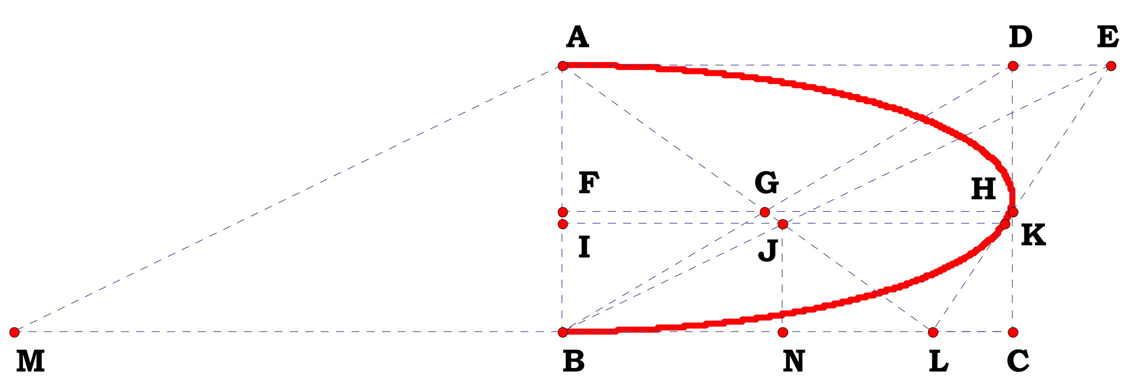
N_1 = 0.37535

N_2 = 0.66604

$N_1 \cdot N_2 - N_0^2$ = 0.00000

$BF := AB - AF \quad FG := \dfrac{BC \cdot BF}{AB} \quad BL := \dfrac{FG \cdot AB}{AF} \quad CL := BC - BL$

$DE := \dfrac{CL \cdot AF}{BF} \quad AE := AD + DE \quad \sqrt{BL \cdot AE} - AD = \blacksquare \quad JN := \dfrac{AB \cdot BL}{AE + BL}$



After one acquires the equation, even if one put 0 in for the unit of the geometry, the equation remains the same for a circle and ellipse. This means, that no matter how one envision the circle, larger than largest or not at all its definition remains the same.

$BN := \dfrac{AE \cdot JN}{AB} \quad IK := 2 \cdot BN$

$$IK - \frac{2 \cdot N_3 \cdot N_2 - 2 \cdot N_2}{{N_3}^2 - 2 \cdot N_3 + 2} = \blacksquare \qquad IK = \blacksquare$$

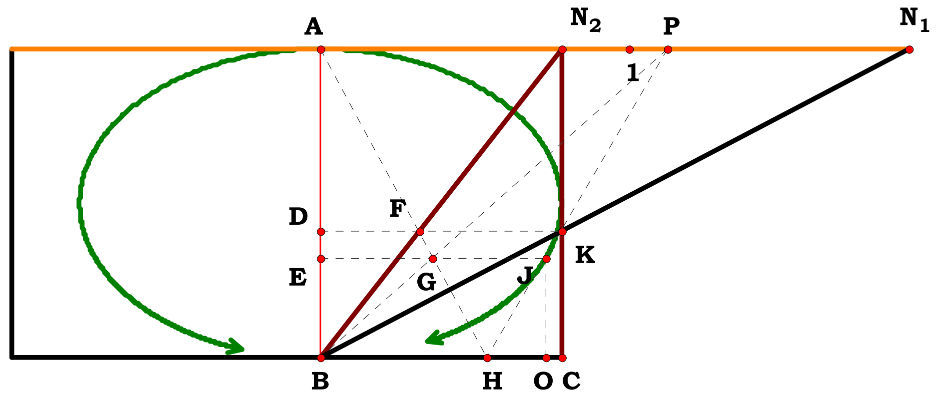
$AB := 1$

Given:

$N_1 := 2.75805$

$N_2 := .79464$

Find EJ.



The Curve of the Equation:

Expressing a circle as a linear function.

Descriptions:

$AD := AB - \frac{N_2}{N_1}$ $\quad DF := N_2 \cdot (AB - AD)$ $\quad BH := \frac{DF}{AD}$ $\quad CH := N_2 - BH$

$CK := AB - AD$ $\quad PN_2 := \frac{CH \cdot AD}{CK}$ $\quad AP := N_2 + PN_2$

$AE := \frac{AB \cdot AP}{AP + BH}$ $\quad BE := AB - AE$ $\quad HO := \frac{CH \cdot BE}{CK}$ $\quad EJ := BH + HO$

Definitions:

$$EJ - \frac{2 \cdot N_2^{\,2} \cdot (N_1 - N_2)}{N_1^{\,2} - 2 \cdot N_1 \cdot N_2 + 2 \cdot N_2^{\,2}} = 0$$

$$\sqrt{AP \cdot BH} - N_2 = 0 \qquad AP - (N_1 - N_2) = 0 \qquad BH - \frac{N_2^{\,2}}{N_1 - N_2} = 0$$

$$\frac{N_2 \cdot \left[EJ + N_2 + \frac{N_1 - 1}{\sqrt{(N_1 - 1)^2}} \cdot \sqrt{N_2^{\,2} - EJ^2}\right]}{EJ} - N_1 = 0$$

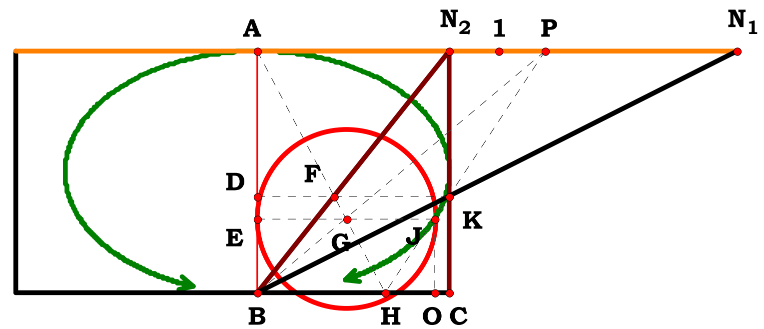
Notice that EJ is not expressed as a root function.

Given:

$EJ = 0.552689$

$AE = 0.859253$

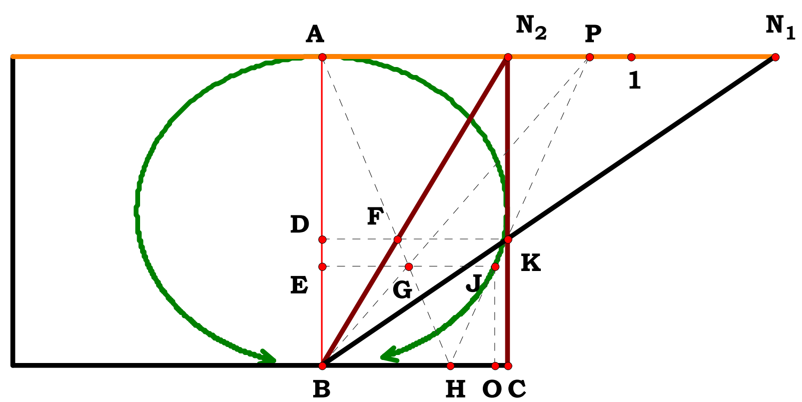
Find N_1 an N_2:



$$BH - \frac{EJ}{2 \cdot AE} = 0 \qquad BE - (AB - AE) = 0 \qquad AP - \left(BH + \frac{EJ - BH}{BE}\right) = 0$$

$$N_2 - \sqrt{BH \cdot AP} = 0 \qquad CK - BE \cdot \frac{N_2 - BH}{EJ - BH} = 0 \qquad N_1 - \frac{N_2}{CK} = 0 \qquad CK - \frac{N_2}{N_1} = 0$$

Given N_1 an N_2: find EJ.



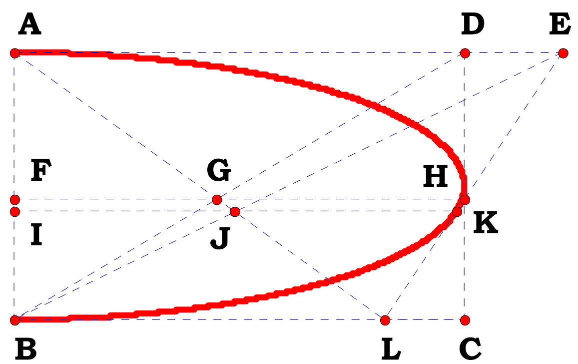
Descriptions:

$$AP - (N_1 - N_2) = 0 \qquad CK - \frac{N_2}{N_1} = 0 \qquad CH - \left(\frac{AP - N_2}{AB - CK}\right) \cdot CK = 0 \qquad BH - (N_2 - CH) = 0$$

$$EJ - 2\frac{AP \cdot BH}{AP + BH} = 0 \qquad EJ - 2\frac{(N_1 - N_2) \cdot \frac{N_2^{\,2}}{N_1 - N_2}}{N_1 - N_2 + \frac{N_2^{\,2}}{N_1 - N_2}} = 0 \qquad EJ - \frac{2 \cdot N_2^{\,2} \cdot (N_1 - N_2)}{N_1^{\,2} - 2 \cdot N_1 \cdot N_2 + 2 \cdot N_2^{\,2}} = 0$$

Notice that IK is not expressed as a root function.

$$\frac{N_2 \cdot \left(2 \cdot N_3 - 2 \cdot N_3^{\,2}\right)}{2 \cdot N_3^{\,2} - 2 \cdot N_3 + 1} = 0.779338$$



$N_1 \equiv 3.51896$

$N_2 \equiv 5.95313$

$N_3 \equiv .54930$

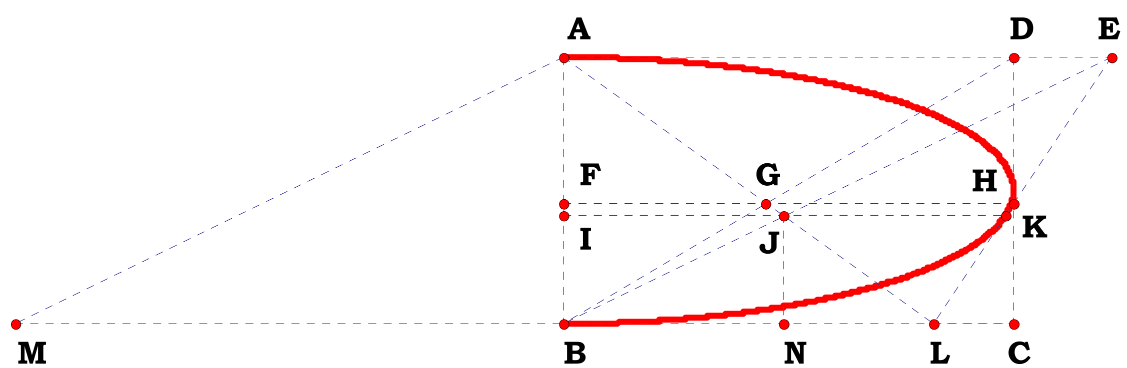
$AB := N_1$

$AD := N_2 \quad BC := AD$

$AF := N_3 \cdot AB$

$BF := AB - AF \quad FG := \dfrac{BC \cdot BF}{AB} \quad BL := \dfrac{FG \cdot AB}{AF} \quad CL := BC - BL$

$DE := \dfrac{CL \cdot AF}{BF} \quad AE := AD + DE \quad \sqrt{BL \cdot AE} - AD = 0 \quad JN := \dfrac{AB \cdot BL}{(AE + BL)}$



$BN := \dfrac{AE \cdot JN}{AB} \quad IK := 2 \cdot BN \quad IK - \dfrac{N_2 \cdot \left(2 \cdot N_3 - 2 \cdot N_3^{\,2}\right)}{2 \cdot N_3^{\,2} - 2 \cdot N_3 + 1} = 0$

AB = 3.93700 cm

BC = 1.96850 cm

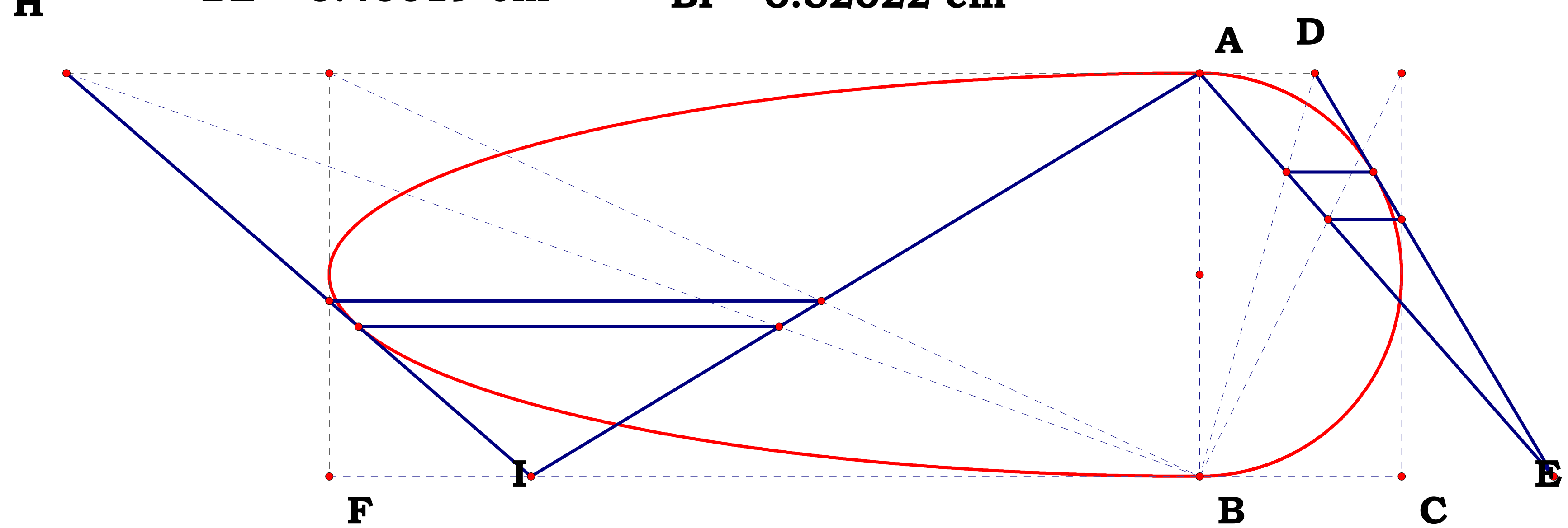
AD = 1.12312 cm

BE = 3.45019 cm

BF = 8.48783 cm

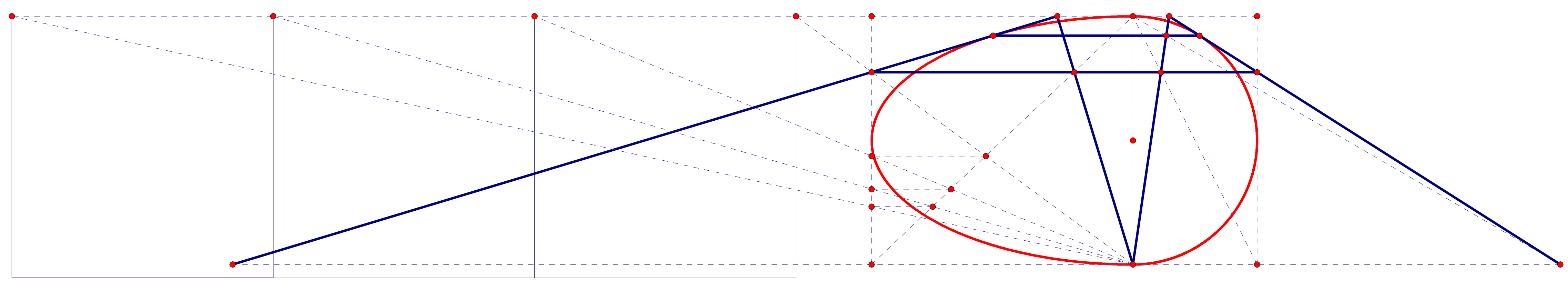
AH = 11.04921 cm

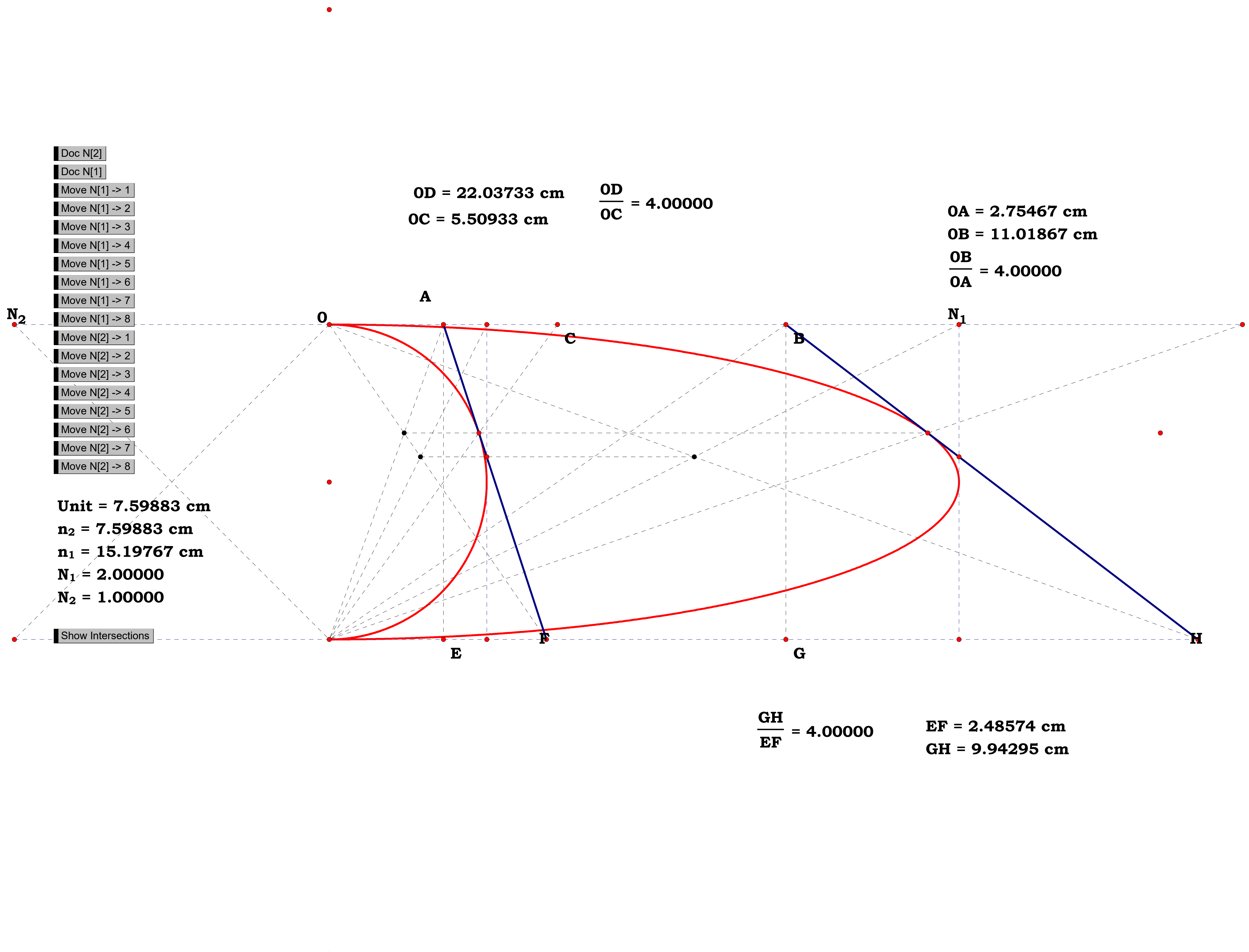
BI = 6.52022 cm



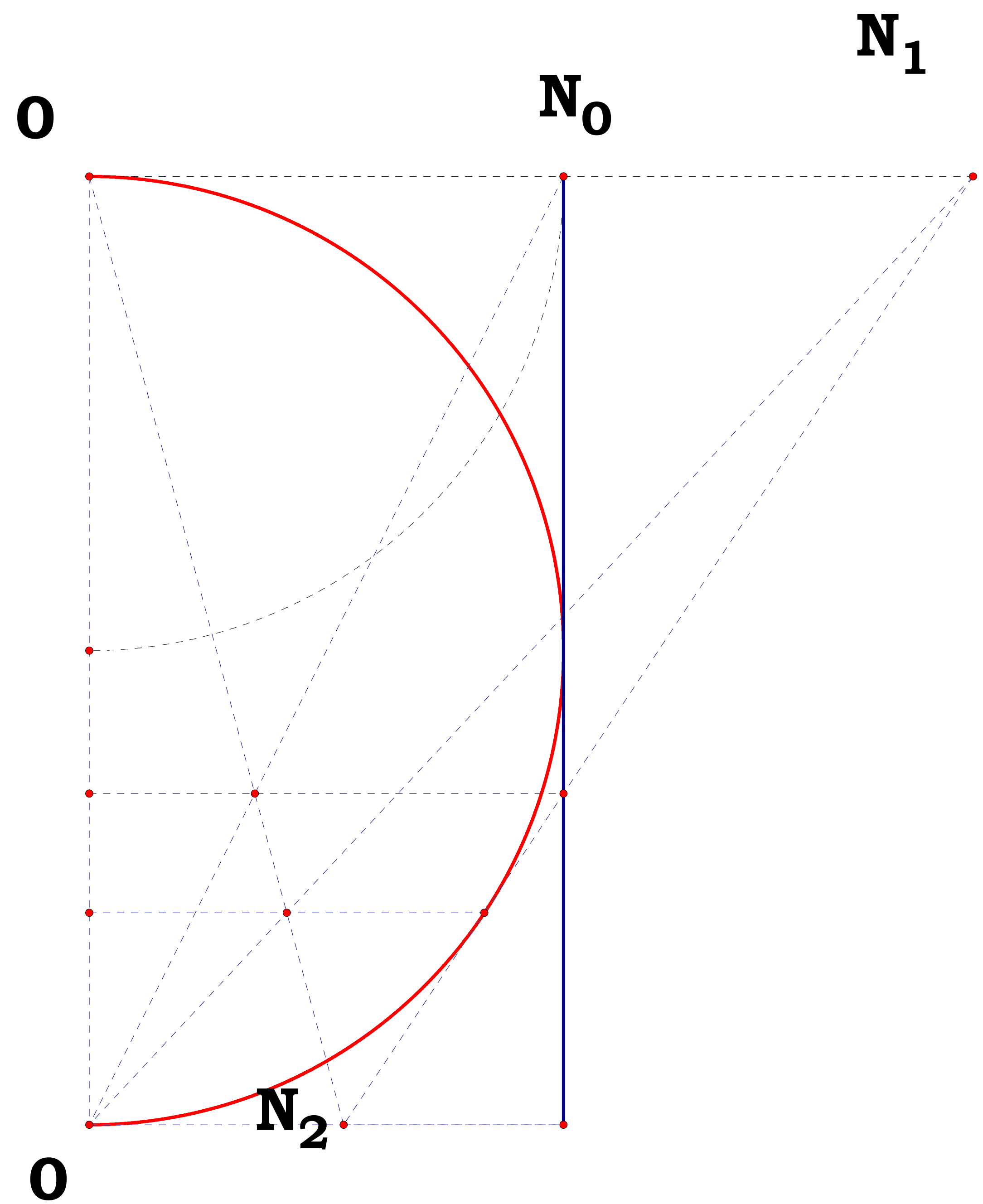
$\sqrt{BE \cdot AD} - BC = 0.00000$ **cm**

$\sqrt{AH \cdot BI} - BF = 0.00000$ **cm**





Doc N[2]
Doc N[1]
Move N[1] -> 1
Move N[1] -> 2
Move N[1] -> 3
Move N[1] -> 4
Move N[1] -> 5
Move N[1] -> 6
Move N[1] -> 7
Move N[1] -> 8
Move N[2] -> 1
Move N[2] -> 2
Move N[2] -> 3
Move N[2] -> 4
Move N[2] -> 5
Move N[2] -> 6
Move N[2] -> 7
Move N[2] -> 8
Show Intersections
0D = 22.03733 cm
0C = 5.50933 cm
0D/0C = 4.00000
0A = 2.75467 cm
0B = 11.01867 cm
0B/0A = 4.00000
Unit = 7.59883 cm
n2 = 7.59883 cm
n1 = 15.19767 cm
N1 = 2.00000
N2 = 1.00000
GH/EF = 4.00000
EF = 2.48574 cm
GH = 9.94295 cm
N2
0
A
C
B
N1
D
E
F
G
H



$OO = 4.61433$ cm

$N_0O = 2.30717$ cm

$N_1O = 4.30019$ cm

$N_2O = 1.23786$ cm

$N_1O \cdot N_2O - N_0O^2 = 0.00000 \text{ cm}^2$

$N_0 = 0.50000$

$N_1 = 0.93192$

$N_2 = 0.26826$

$N_1 \cdot N_2 - N_0{}^2 = 0.00000$

$\sqrt{N_1 \cdot N_2} - N_0 = 0.00000$

One is used to taking the equation for the circle from the inside. Today, however, it is going for a walk in the sunshine!

$OO = 2.34950$ cm

$N_0O = 10.11767$ cm

$N_1O = 7.59534$ cm

$N_2O = 13.47764$ cm

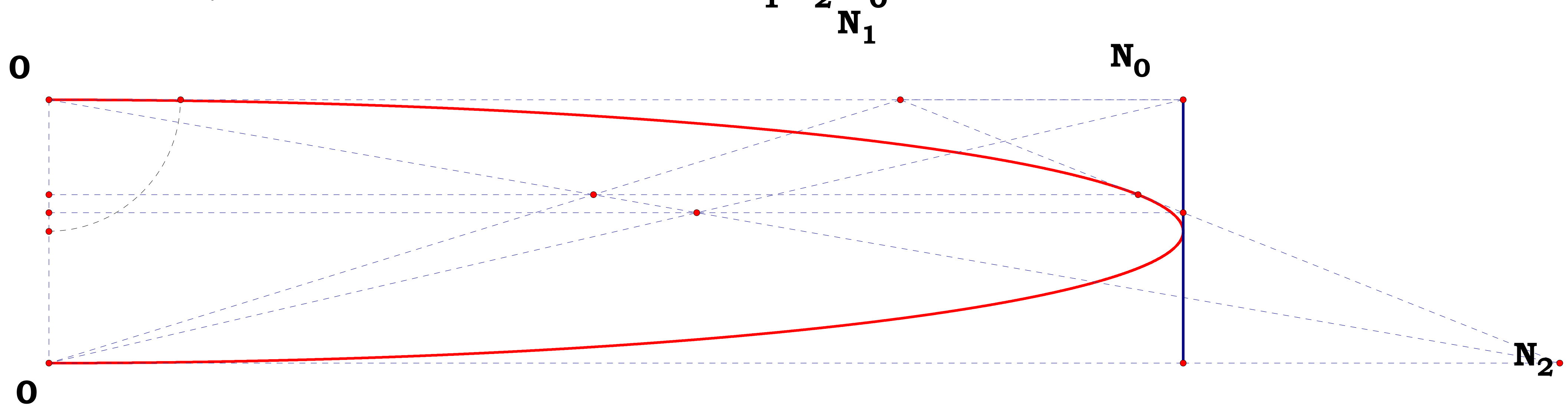
$N_1O \cdot N_2O - N_0O^2 = 0.00000 \text{ cm}^2$

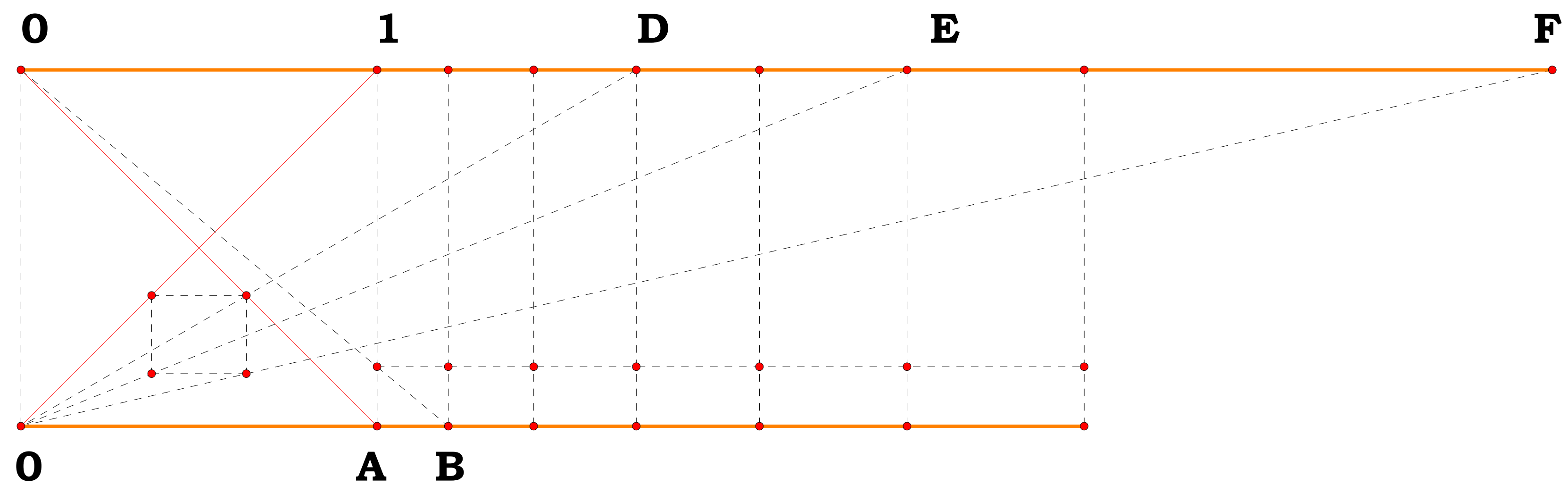
$N_0 = 4.30631$

$N_1 = 3.23275$

$N_2 = 5.73638$

$N_1 \cdot N_2 - N_0^2 = 0.00000$



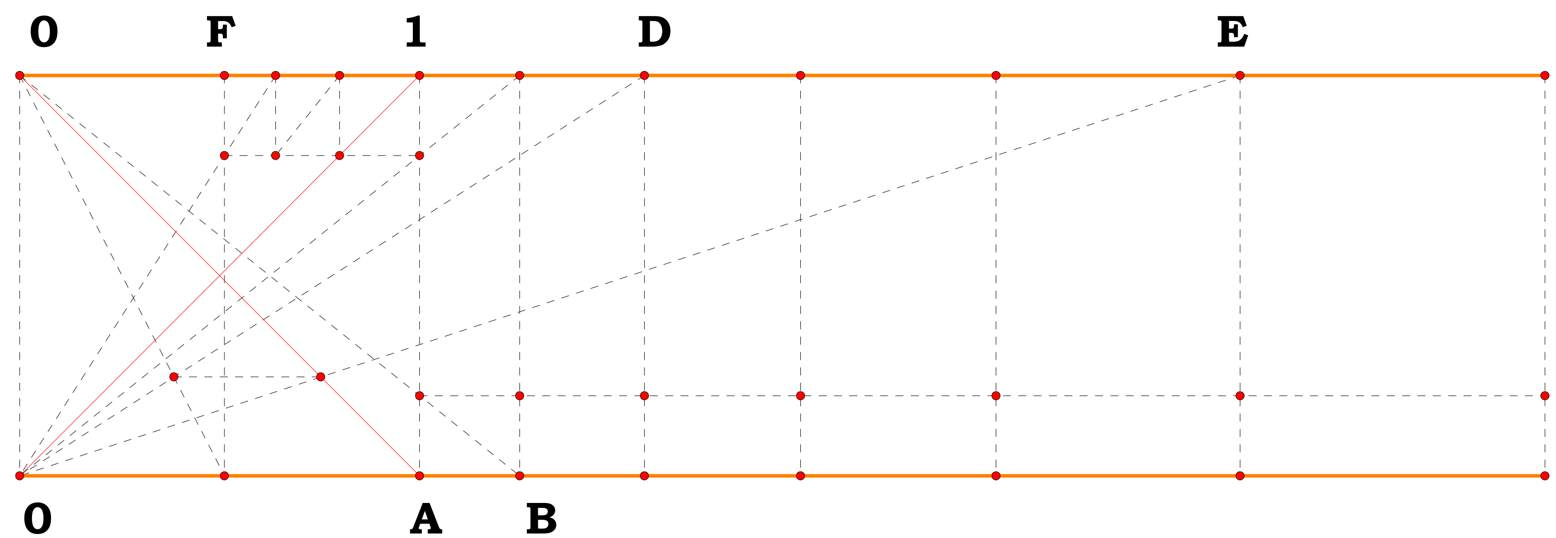


$B = 1.20000$

$D = 1.72800$ $\quad B^3 = 1.72800$

$E = 2.48832$ $\quad B^5 = 2.48832$

$F = 4.29982$ $\quad B^{3+5} = 4.29982$



$B = 1.25000$

$D = 1.56250$

$E = 3.05176$

$F = 0.51200$

$B^2 = 1.56250$

$B^5 = 3.05176$

$B^{-3} = 0.51200$

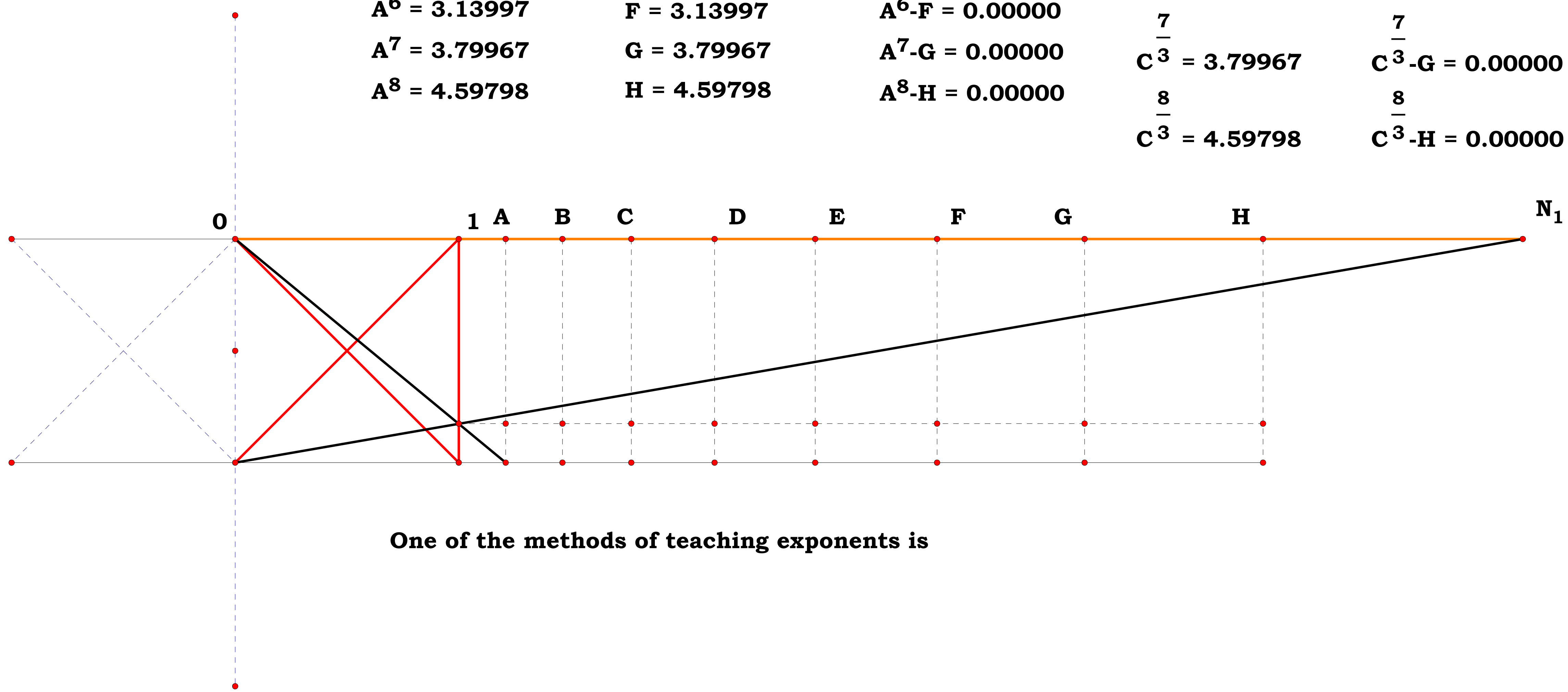
$B^{2-5} = 0.51200$

$$\frac{1}{B^3} = 0.51200$$

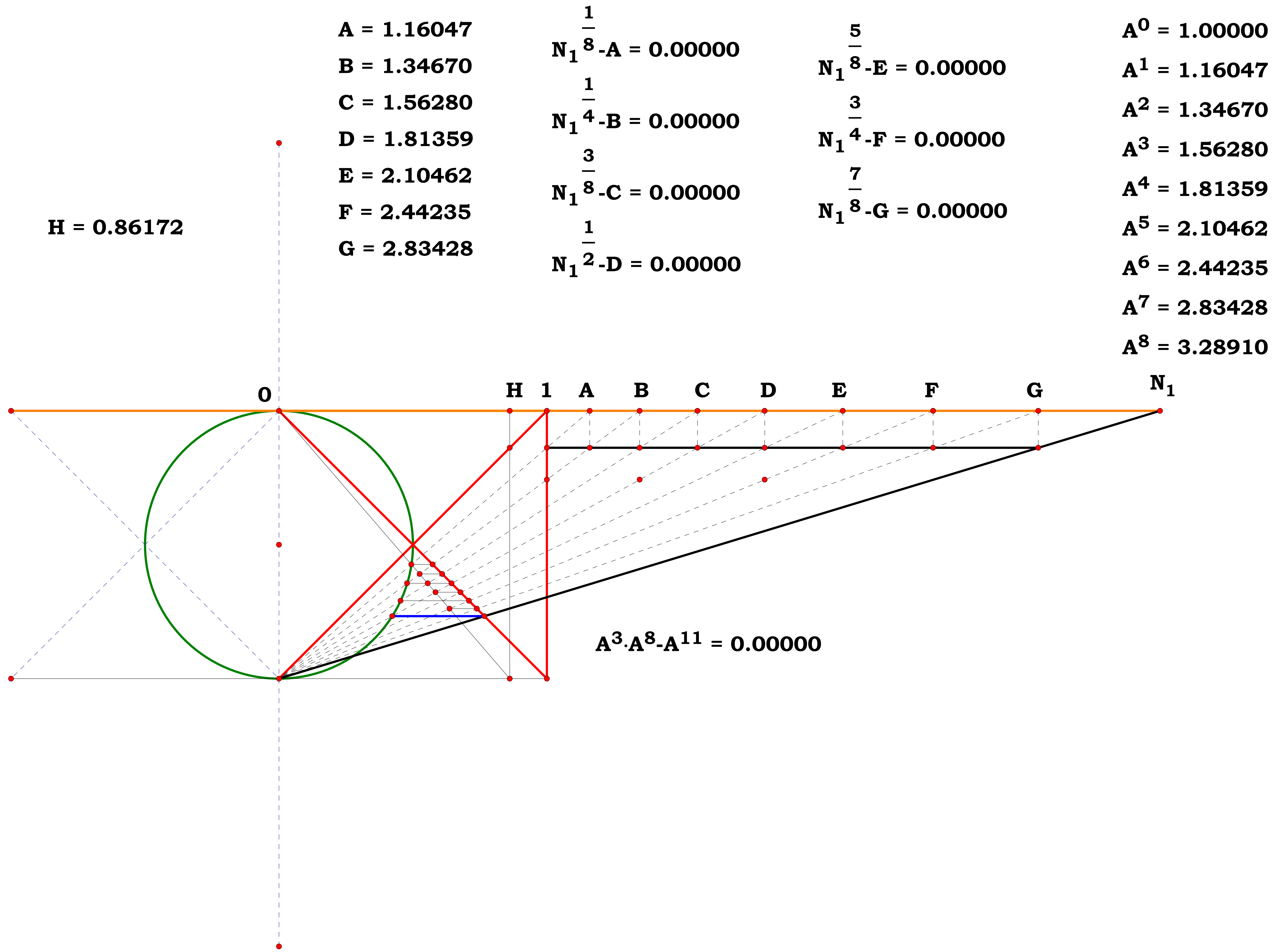
$$\frac{N_1}{N_1-1} = 1.21010$$

$$\frac{N_1}{A} = 4.75966$$

	$A = 1.21010$				
$A^2 = 1.46434$	$B = 1.46434$	$A^2\text{-}B = 0.00000$	$C^{\frac{2}{3}} = 1.46434$	$C^{\frac{2}{3}}\text{-}B = 0.00000$	
$A^3 = 1.77200$	$C = 1.77200$	$A^3\text{-}C = 0.00000$	$C^{\frac{4}{3}} = 2.14429$	$C^{\frac{4}{3}}\text{-}D = 0.00000$	
$A^4 = 2.14429$	$D = 2.14429$	$A^4\text{-}D = 0.00000$	$C^{\frac{5}{3}} = 2.59480$	$C^{\frac{5}{3}}\text{-}E = 0.00000$	$E^{\frac{4}{5}}\text{-}D = 0.00000$
$A^5 = 2.59480$	$E = 2.59480$	$A^5\text{-}E = 0.00000$	$C^{\frac{7}{3}} = 3.79967$	$C^{\frac{7}{3}}\text{-}G = 0.00000$	
$A^6 = 3.13997$	$F = 3.13997$	$A^6\text{-}F = 0.00000$	$C^{\frac{8}{3}} = 4.59798$	$C^{\frac{8}{3}}\text{-}H = 0.00000$	
$A^7 = 3.79967$	$G = 3.79967$	$A^7\text{-}G = 0.00000$			
$A^8 = 4.59798$	$H = 4.59798$	$A^8\text{-}H = 0.00000$			



One of the methods of teaching exponents is



A = 1.16047
B = 1.34670
C = 1.56280
D = 1.81359
E = 2.10462
F = 2.44235
G = 2.83428
$N_1^{\frac{1}{8}}$-A = 0.00000
$N_1^{\frac{1}{4}}$-B = 0.00000
$N_1^{\frac{3}{8}}$-C = 0.00000
$N_1^{\frac{1}{2}}$-D = 0.00000
$N_1^{\frac{5}{8}}$-E = 0.00000
$N_1^{\frac{3}{4}}$-F = 0.00000
$N_1^{\frac{7}{8}}$-G = 0.00000
A^0 = 1.00000
A^1 = 1.16047
A^2 = 1.34670
A^3 = 1.56280
A^4 = 1.81359
A^5 = 2.10462
A^6 = 2.44235
A^7 = 2.83428
A^8 = 3.28910
H = 0.86172
0
H
1
A
B
C
D
E
F
G
N_1
$A^3 \cdot A^8 - A^{11}$ = 0.00000

$N_1^{0} = 1.00000$

$N_1^{1} = 1.50898$

$N_1^{-3} = 0.29104$

$N_1^{-2} = 0.43917$

$N_1^{-1} = 0.66270$

$N_1^{2} = 2.27703$

$N_1^{3} = 3.43599$

$N_1^{4} = 5.18485$

$R_1 = 0.29104$

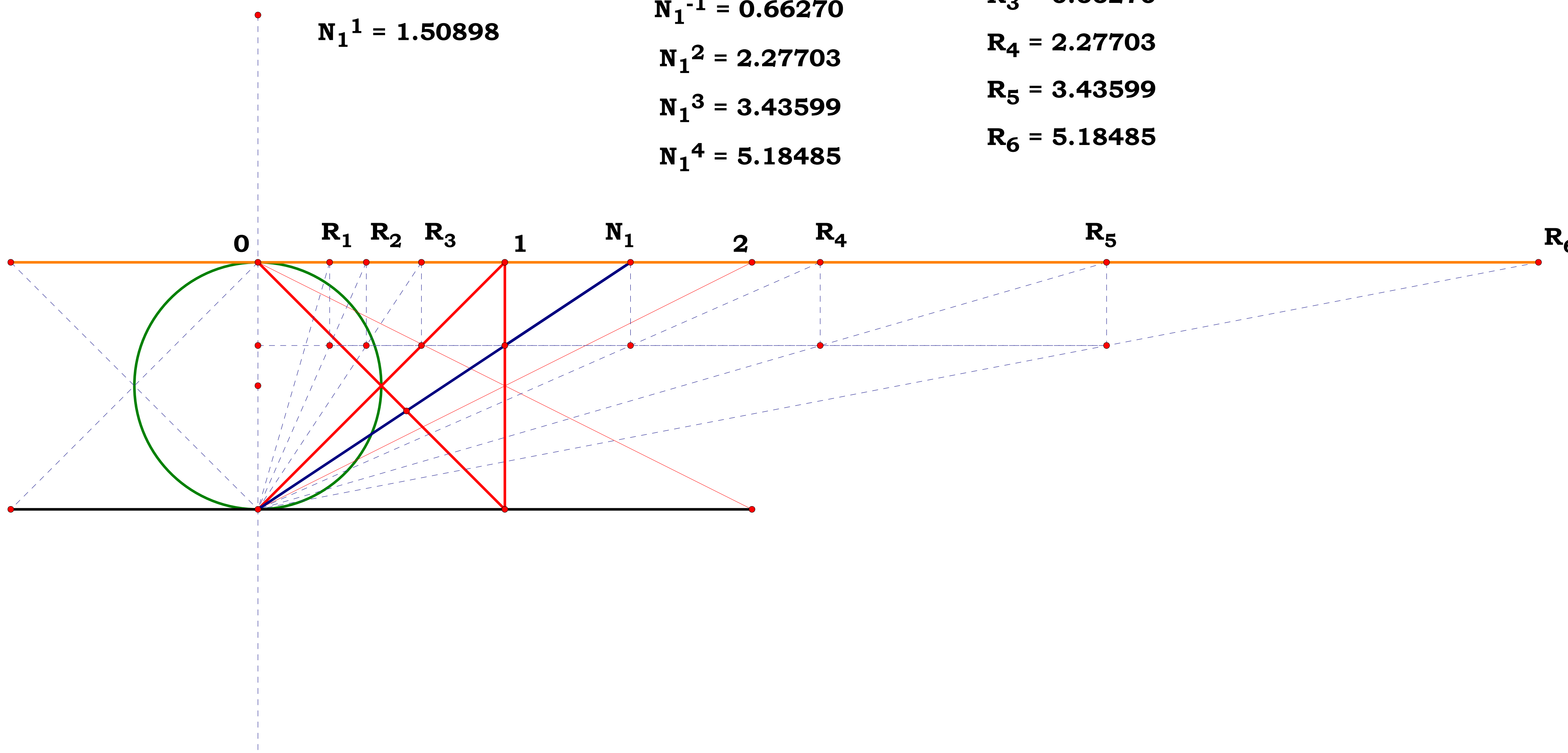
$R_2 = 0.43917$

$R_3 = 0.66270$

$R_4 = 2.27703$

$R_5 = 3.43599$

$R_6 = 5.18485$



$N_1 \cdot 2^{-3} = 0.11077$

$N_1 \cdot 2^{-2} = 0.22154$

$N_1 \cdot 2^{-1} = 0.44309$

$N_1 \cdot 2^{0} = 0.88618$

$N_1 \cdot 2^{1} = 1.77236$

$N_1 \cdot 2^{2} = 3.54472$

$N_1 \cdot 2^{3} = 7.08943$

$R_1 = 0.11077$

$R_2 = 0.22154$

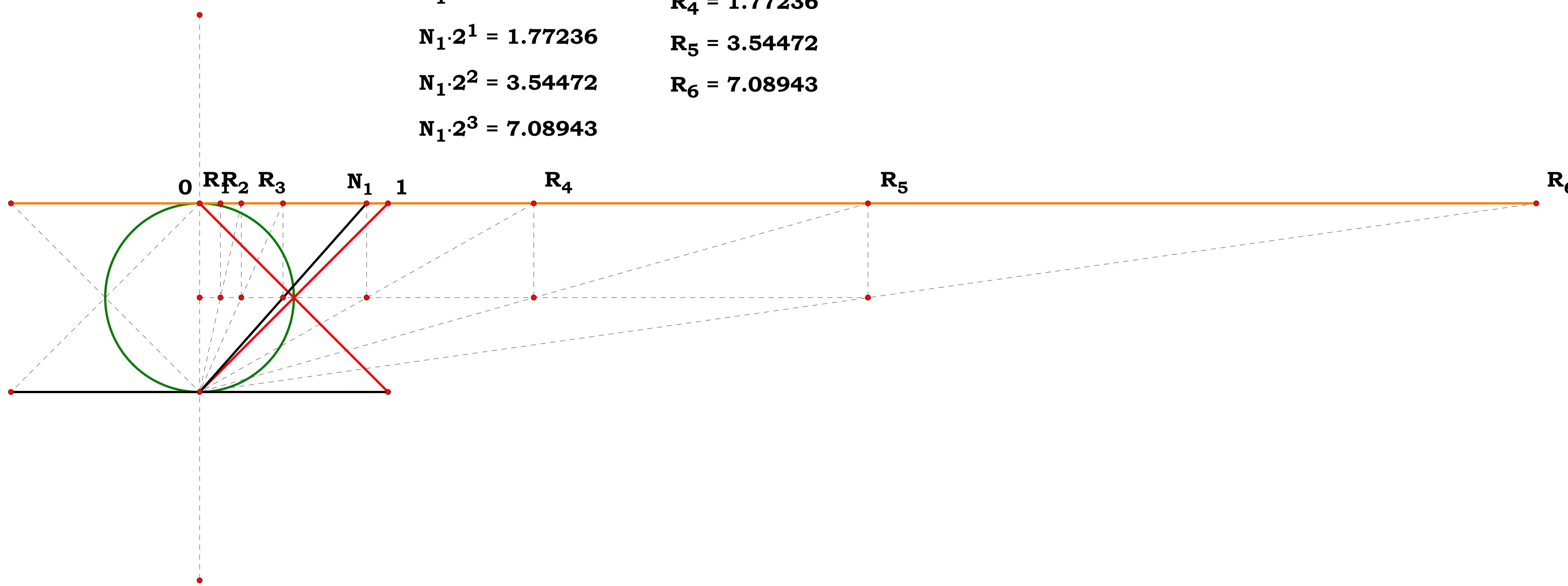
$R_3 = 0.44309$

$R_4 = 1.77236$

$R_5 = 3.54472$

$R_6 = 7.08943$

$N_1 = 0.88618$



$$\frac{0R_2}{0R_1} = 1.42857$$

$$\frac{N_1+1}{N_1} = 1.42857$$

$$\left(\frac{N_1}{R_1}\right)^{\frac{1}{3}} = 1.42857$$

$$\left(\frac{N_1+1}{N_1}\right)^{-1} \cdot N_1 = 1.63333$$

$$\frac{N_1+1}{N_1} \cdot N_1 = 3.33333$$

$$\left(\frac{N_1+1}{N_1}\right)^{2} \cdot N_1 = 4.76190$$

$$R_1 = 0.80033$$

$$R_2 = 1.14333$$

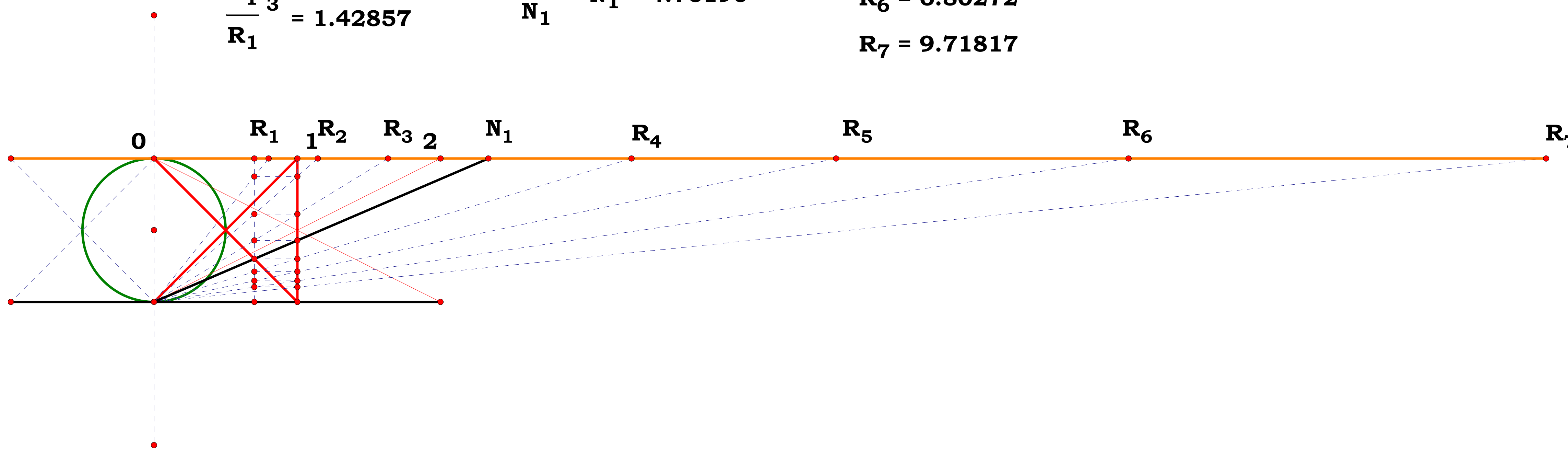
$$R_3 = 1.63333$$

$$R_4 = 3.33333$$

$$R_5 = 4.76190$$

$$R_6 = 6.80272$$

$$R_7 = 9.71817$$



$$\left|N_1^{-4}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_1 = 0.00000$$

$$\left|N_1^{-3}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_2 = 0.00000$$

$$\left|N_1^{-2}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_3 = 0.00000$$

$$\left|N_1^{-1}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_4 = 0.00000$$

$$\left|N_1^{0}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_5 = 0.00000$$

$$\left|N_1^{1}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_6 = 0.00000$$

$$\left|N_1^{2}\cdot\left(N_1+\frac{N_1}{N_1-1}\right)\right|-B_7 = 0.00000$$

$N_1^{0} = 1.00000$

$N_1^{1} = 1.37923$

$N_1^{-3} = 0.38115$

$N_1^{-2} = 0.52569$

$N_1^{-1} = 0.72504$

$N_1^{2} = 1.90227$

$N_1^{3} = 2.62366$

$N_1^{4} = 3.61863$

$R_1 = 0.38115$

$R_2 = 0.52569$

$R_3 = 0.72504$

$R_4 = 1.90227$

$R_5 = 2.62366$

$R_6 = 3.61863$

$B_1 = 1.38620$

$B_2 = 1.91189$

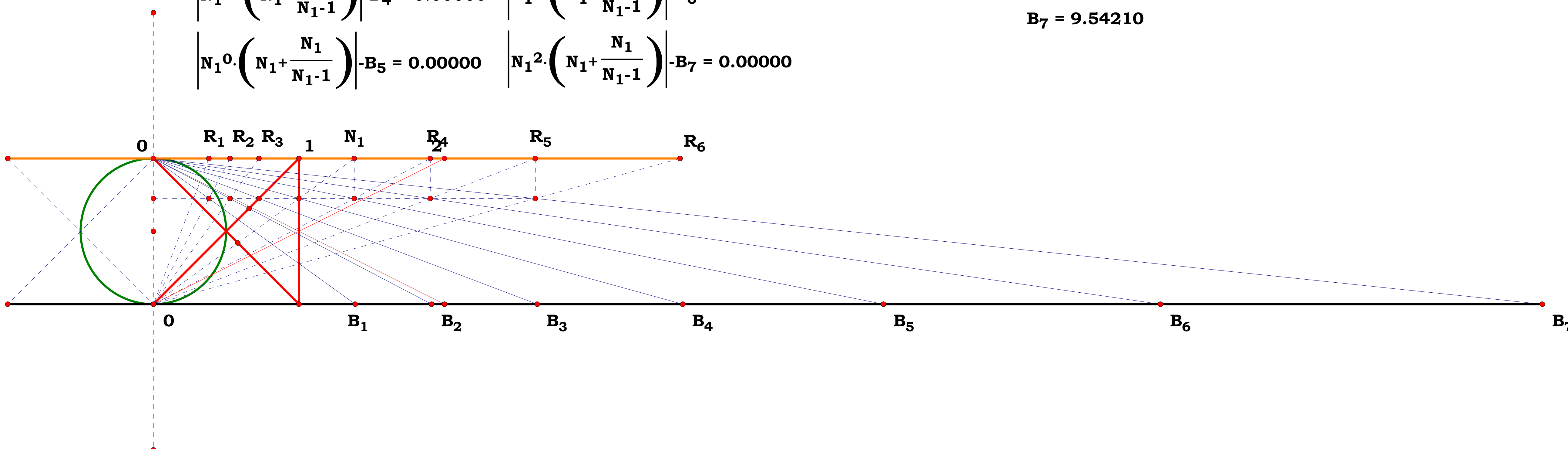
$B_3 = 2.63694$

$B_4 = 3.63694$

$B_5 = 5.01616$

$B_6 = 6.91843$

$B_7 = 9.54210$



$$\frac{R_2}{R_1} = 1.28271$$

$$\frac{N_1^2}{N_1+1} = 1.23153$$

$$N_1 \cdot \frac{N_1^2}{N_1+1}^{-3} - R_1 = 0.11604$$

$$N_1 \cdot \frac{N_1^2}{N_1+1}^{-2} - R_2 = 0.09720$$

$$N_1 \cdot \frac{N_1^2}{N_1+1}^{-1} - R_3 = 0.06107$$

$$N_1 \cdot \frac{N_1^2}{N_1+1} - R_4 = -0.09647$$

$$N_1 \cdot \frac{N_1^2}{N_1+1}^{2} - R_5 = -0.24255$$

$$N_1 \cdot \frac{N_1^2}{N_1+1}^{3} - R_6 = -0.45743$$

$R_1 = 0.89311$

$R_2 = 1.14560$

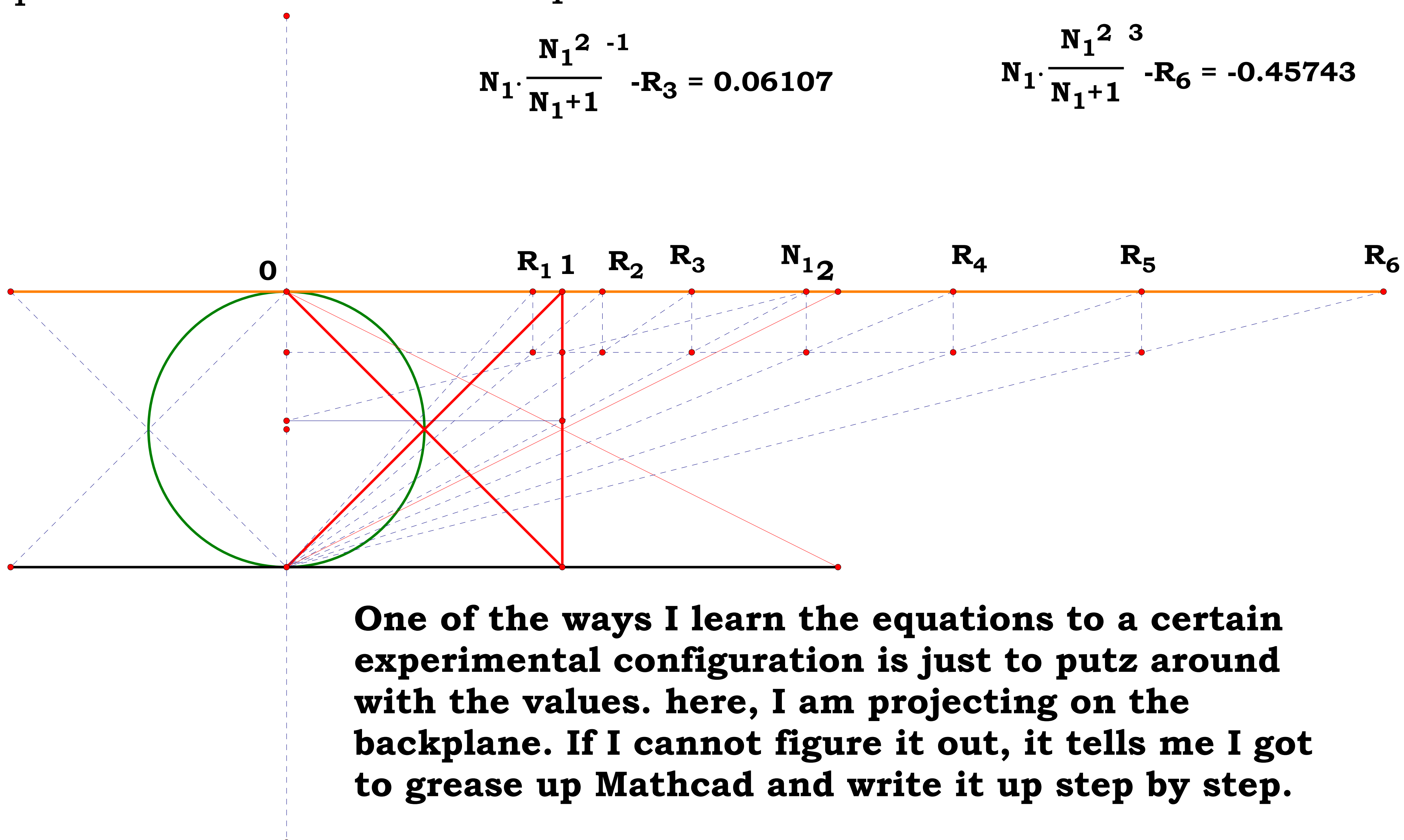
$R_3 = 1.46947$

$N_1 = 1.88489$

$R_4 = 2.41776$

$R_5 = 3.10128$

$R_6 = 3.97803$



One of the ways I learn the equations to a certain experimental configuration is just to putz around with the values. here, I am projecting on the backplane. If I cannot figure it out, it tells me I got to grease up Mathcad and write it up step by step.

$$\frac{N_2+1}{N_2} = 1.25000$$

$$N_1 \cdot {\frac{N_2+1}{N_2}}^{-3} - R_1 = 0.00000$$

$$N_1 \cdot {\frac{N_2+1}{N_2}}^{-2} - R_2 = 0.00000$$

$$N_1 \cdot {\frac{N_2+1}{N_2}}^{-1} - R_3 = 0.00000$$

$$N_1 \cdot \frac{N_2+1}{N_2} - R_4 = 0.00000$$

$$N_1 \cdot {\frac{N_2+1}{N_2}}^{2} - R_5 = 0.00000$$

$$N_1 \cdot {\frac{N_2+1}{N_2}}^{3} - R_6 = 0.00000$$

$R_1 = 1.53600$

$R_2 = 1.92000$

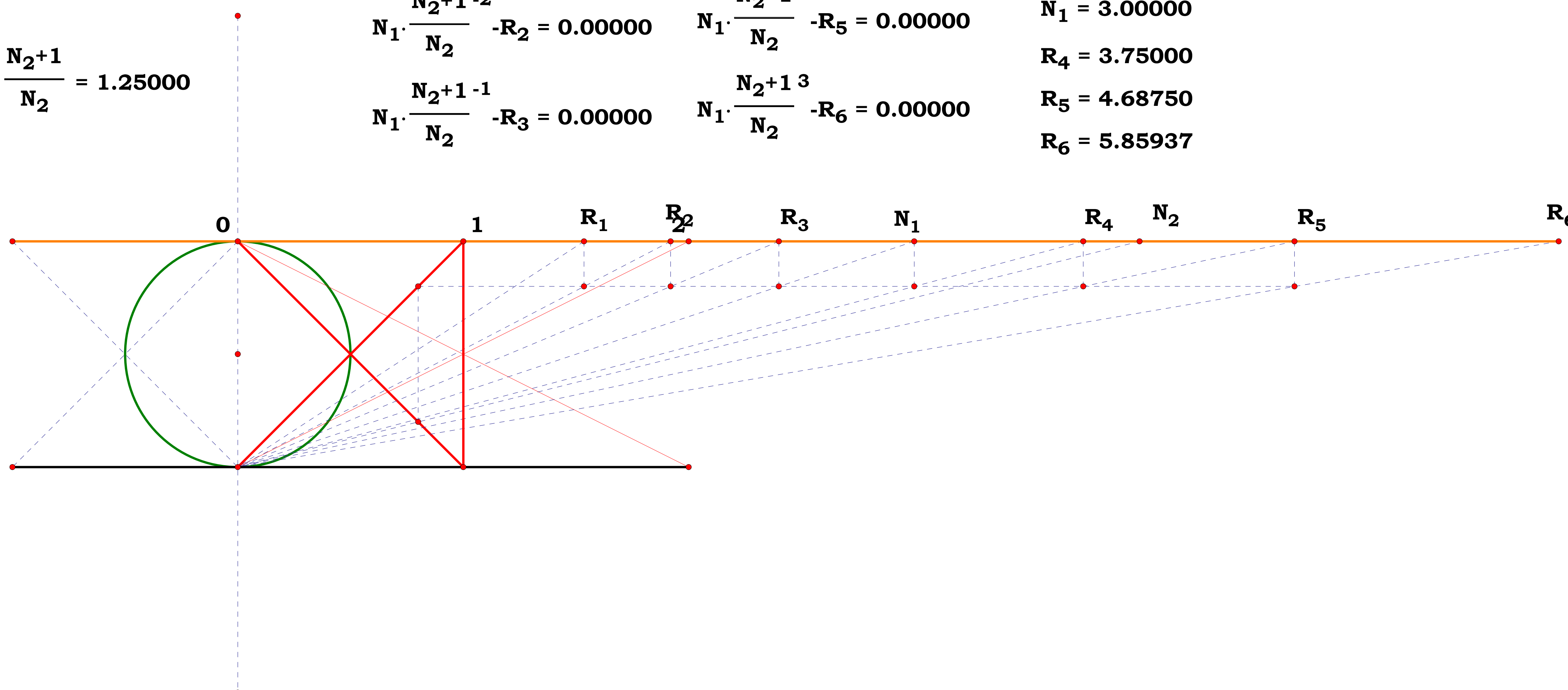
$R_3 = 2.40000$

$N_1 = 3.00000$

$R_4 = 3.75000$

$R_5 = 4.68750$

$R_6 = 5.85937$



$$\frac{N_2 \cdot (N_1+1)}{N_1 \cdot (N_2+1)} = 1.18038$$

$R_3^{1+\frac{1}{3}} = 1.94130$

$R_3^2 = 2.70482$

$R_1 3^{2+\frac{1}{3}} = 3.19272$

$R_3^3 = 4.44843$

$R_3^{2+\frac{2}{3}} = 3.76863$

$R_3^3 = 4.44843$

$R_3^4 = 7.31604$

$R_2^{1.5} = 1.64463$

$R_2^2 = 1.94130$

$R_2^3 = 2.70482$

$R_2^{3.5} = 3.19272$

$R_2^4 = 3.76863$

$R_2^{4.5} = 4.44843$

$R_2^5 = 5.25085$

$R_2^{5.5} = 6.19802$

$R_2^6 = 7.31604$

$R_1^2 = 1.39330$

$R_1^3 = 1.64463$

$R_1^4 = 1.94130$

$R_1^5 = 2.29147$

$R_1^6 = 2.70482$

$R_1^7 = 3.19272$

$R_1^8 = 3.76863$

$R_1^9 = 4.44843$

$R_1^{10} = 5.25085$

$R_1^{11} = 6.19802$

$R_1^{12} = 7.31604$

$R_1 = 1.18038$

$R_2 = 1.39330$

$R_3 = 1.64463$

$R_4 = 1.94130$

$R_5 = 2.29147$

$R_6 = 2.70482$

$R_7 = 3.19272$

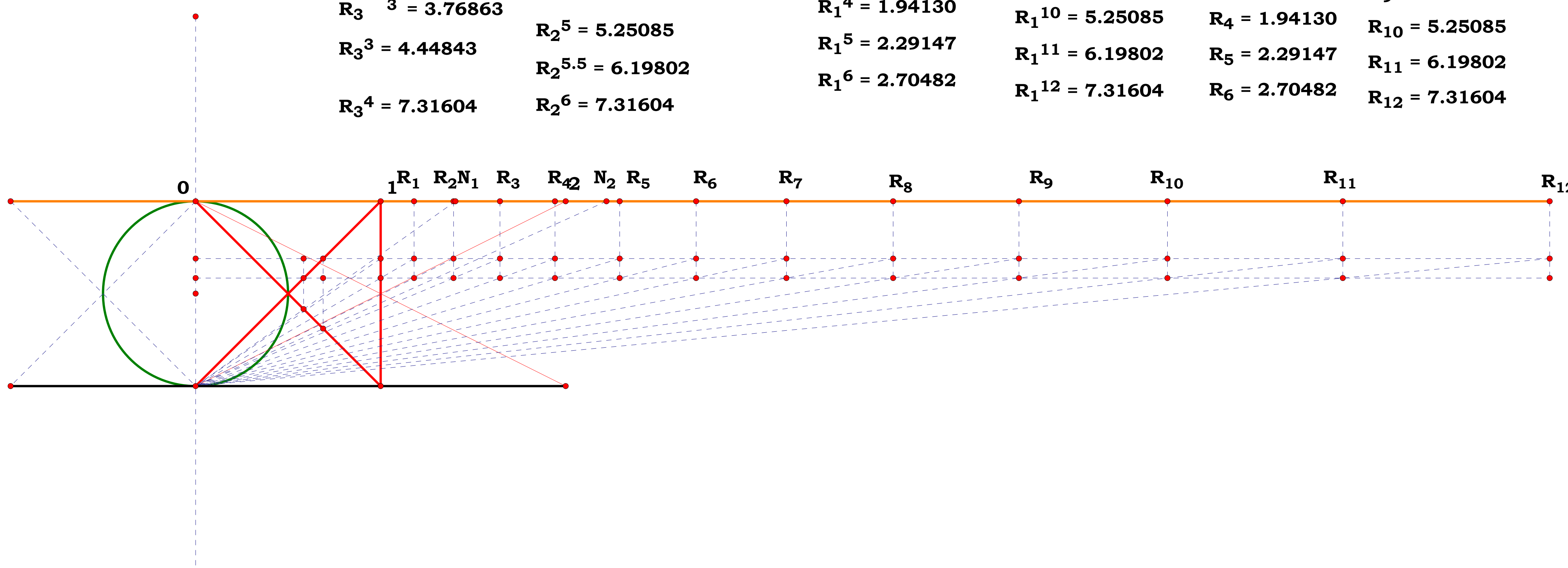
$R_8 = 3.76863$

$R_9 = 4.44843$

$R_{10} = 5.25085$

$R_{11} = 6.19802$

$R_{12} = 7.31604$



$$\frac{N_2}{N_2-1} = 1.63265$$

$$N_1 \cdot \frac{N_2}{N_2-1} - R_1 = 0.00000$$

$$N_1 \cdot {\frac{N_2}{N_2-1}}^{2} - R_2 = 0.00000$$

$$N_1 \cdot {\frac{N_2}{N_2-1}}^{3} - R_3 = 0.00000$$

$$N_1 \cdot {\frac{N_2}{N_2-1}}^{-3} - R_{-3} = 0.00000$$

$$N_1 \cdot {\frac{N_2}{N_2-1}}^{-2} - R_{-2} = 0.00000$$

$$N_1 \cdot {\frac{N_2}{N_2-1}}^{-1} - R_{-1} = 0.00000$$

$R_{-3} = 0.34928$

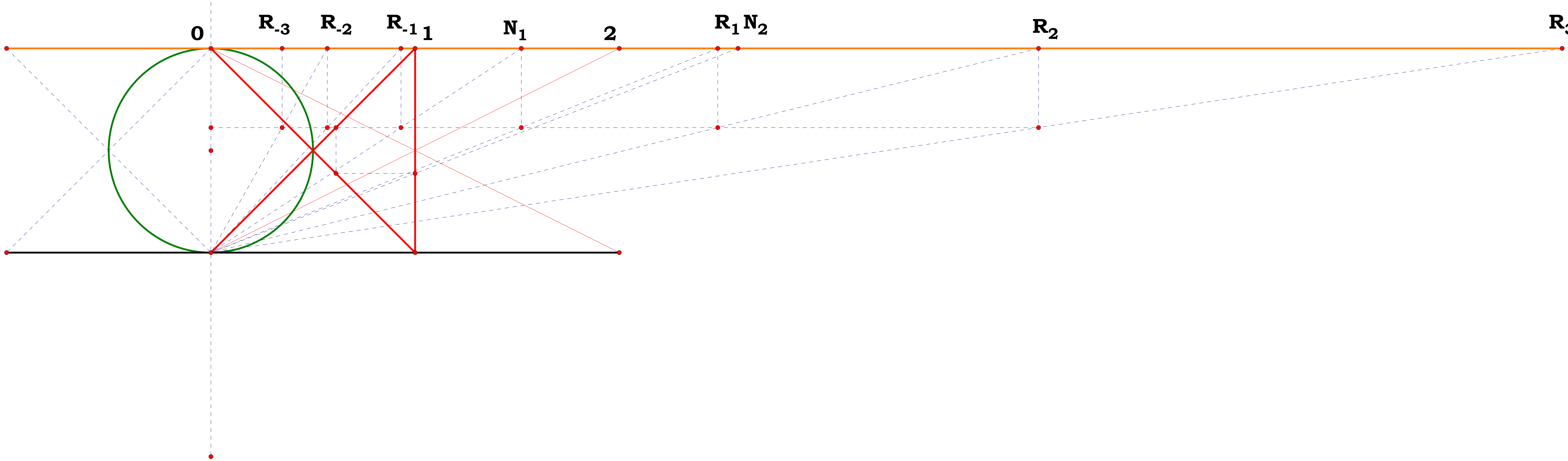
$R_{-2} = 0.57025$

$R_{-1} = 0.93102$

$R_1 = 2.48169$

$R_2 = 4.05174$

$R_3 = 6.61509$



$N_2\text{-}1 = 1.44366$

$(N_2\text{-}1)^{-2} = 0.47981$ $R_{-2} = 0.47981$

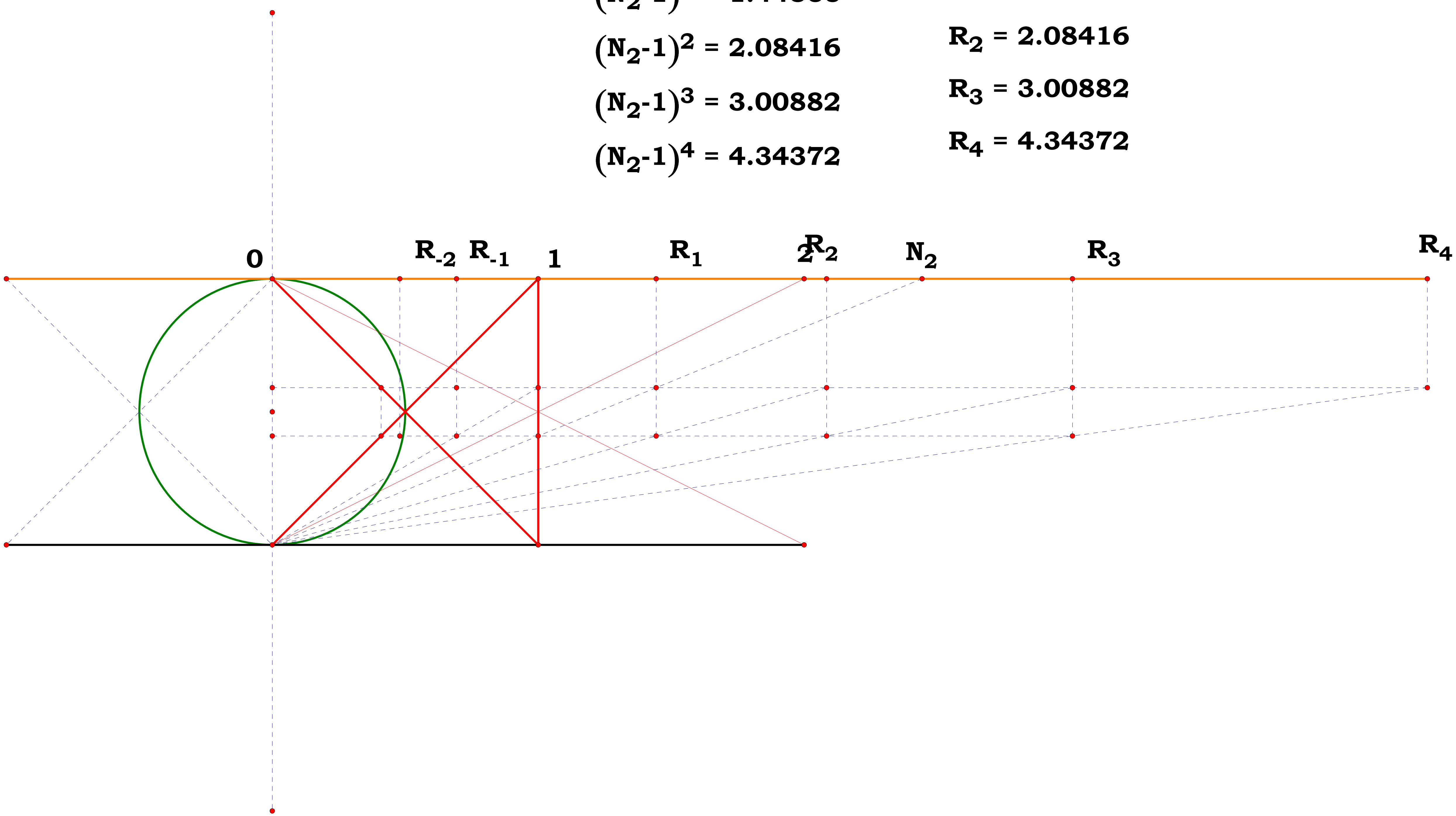
$(N_2\text{-}1)^{-1} = 0.69268$ $R_{-1} = 0.69268$

$(N_2\text{-}1)^{1} = 1.44366$ $R_{1} = 1.44366$

$(N_2\text{-}1)^{2} = 2.08416$ $R_{2} = 2.08416$

$(N_2\text{-}1)^{3} = 3.00882$ $R_{3} = 3.00882$

$(N_2\text{-}1)^{4} = 4.34372$ $R_{4} = 4.34372$



$N_1^{-2} - R_{-2} = 0.00000$ $\quad R_{-2} = 0.49470$

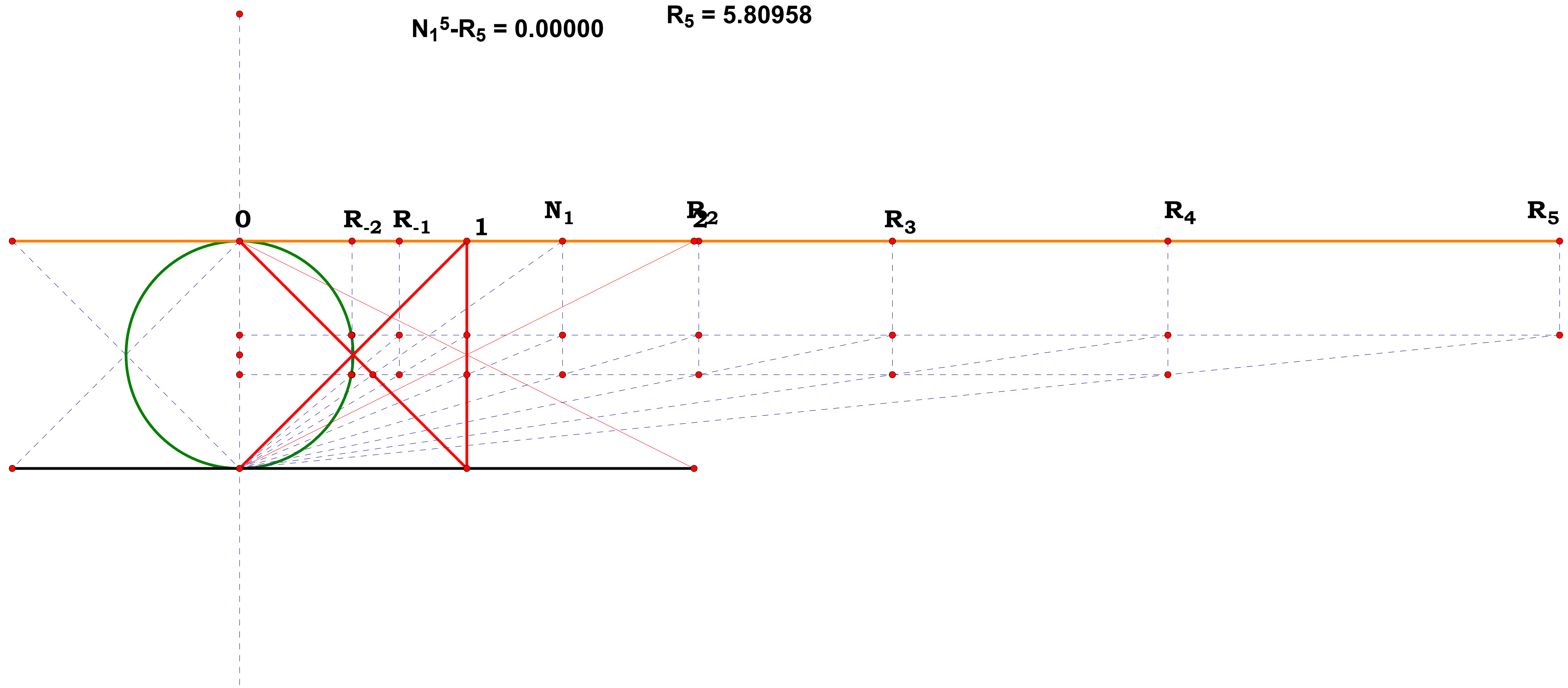
$N_1^{-1} - R_{-1} = 0.00000$ $\quad R_{-1} = 0.70335$

$N_1^{2} - R_{2} = 0.00000$ $\quad R_{2} = 2.02143$

$N_1^{3} - R_{3} = 0.00000$ $\quad R_{3} = 2.87400$

$N_1^{4} - R_{4} = 0.00000$ $\quad R_{4} = 4.08616$

$N_1^{5} - R_{5} = 0.00000$ $\quad R_{5} = 5.80958$



$N_2 = 3.00000$

$N_1 = 4.00000$

$\frac{N_2}{N_1} = 0.75000$

$\frac{N_2}{N_1}^{-3} - R_{-3} = 0.00000$

$\frac{N_2}{N_1}^{-2} - R_{-2} = 0.00000$

$\frac{N_2}{N_1}^{-1} - R_{-1} = 0.00000$

$\frac{N_2}{N_1}^{1} - R_1 = 0.00000$

$\frac{N_2}{N_1}^{2} - R_2 = 0.00000$

$\frac{N_2}{N_1}^{3} - R_3 = 0.00000$

$\frac{N_2}{N_1}^{4} - R_4 = 0.00000$

$R_{-3} = 2.37037$

$R_{-2} = 1.77778$

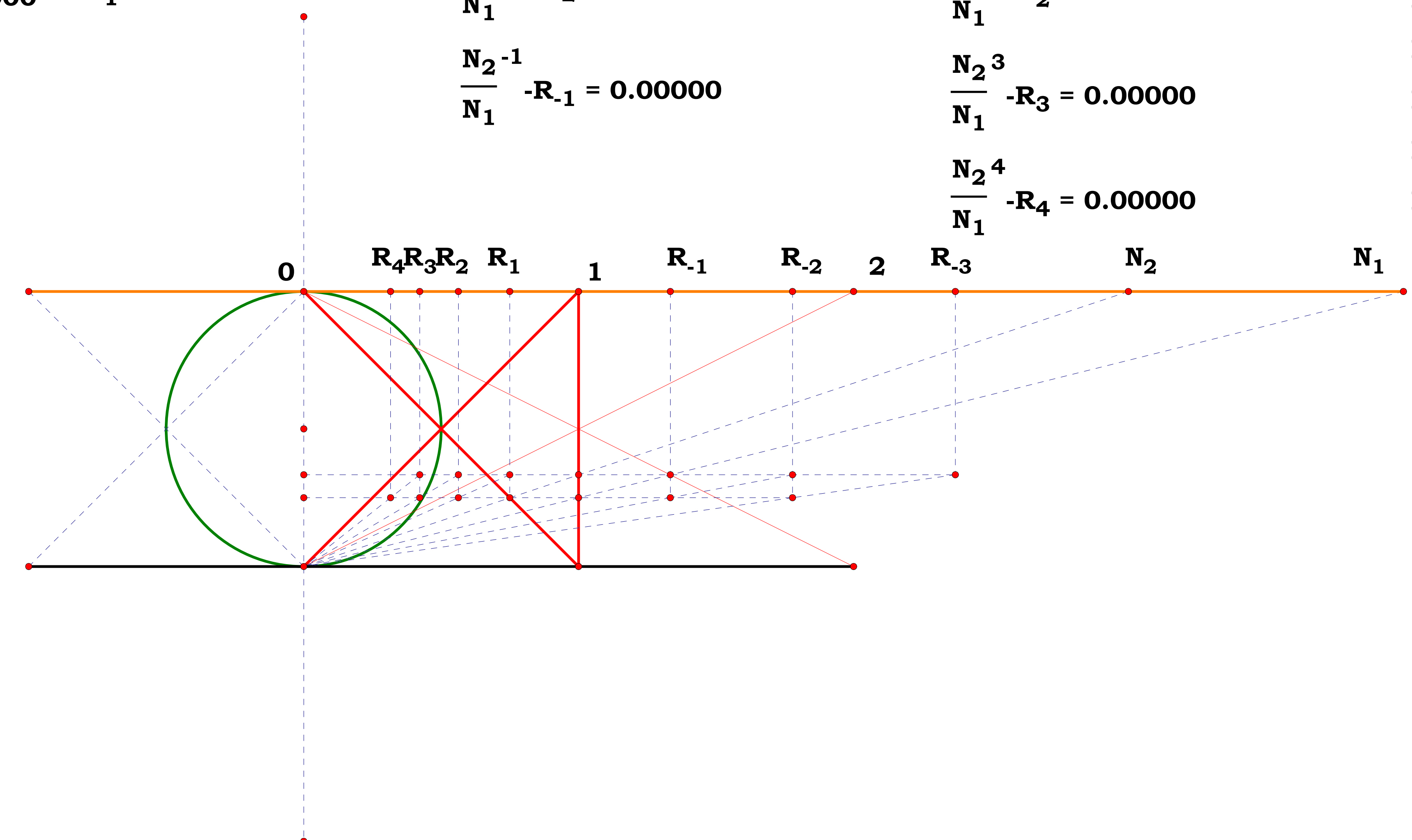
$R_{-1} = 1.33333$

$R_1 = 0.75000$

$R_2 = 0.56250$

$R_3 = 0.42188$

$R_4 = 0.31641$



$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{5} - R_5 = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{1} - R_1 = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{2} - R_2 = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{3} - R_3 = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{4} - R_4 = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{-4} - R_{-4} = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{-3} - R_{-3} = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{-2} - R_{-2} = 0.00000$$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}}^{-1} - R_{-1} = 0.00000$$

$R_{-4} = 0.56250$

$R_{-3} = 0.64952$

$R_{-2} = 0.75000$

$R_{-1} = 0.86603$

$R_1 = 1.15470$

$R_2 = 1.33333$

$R_3 = 1.53960$

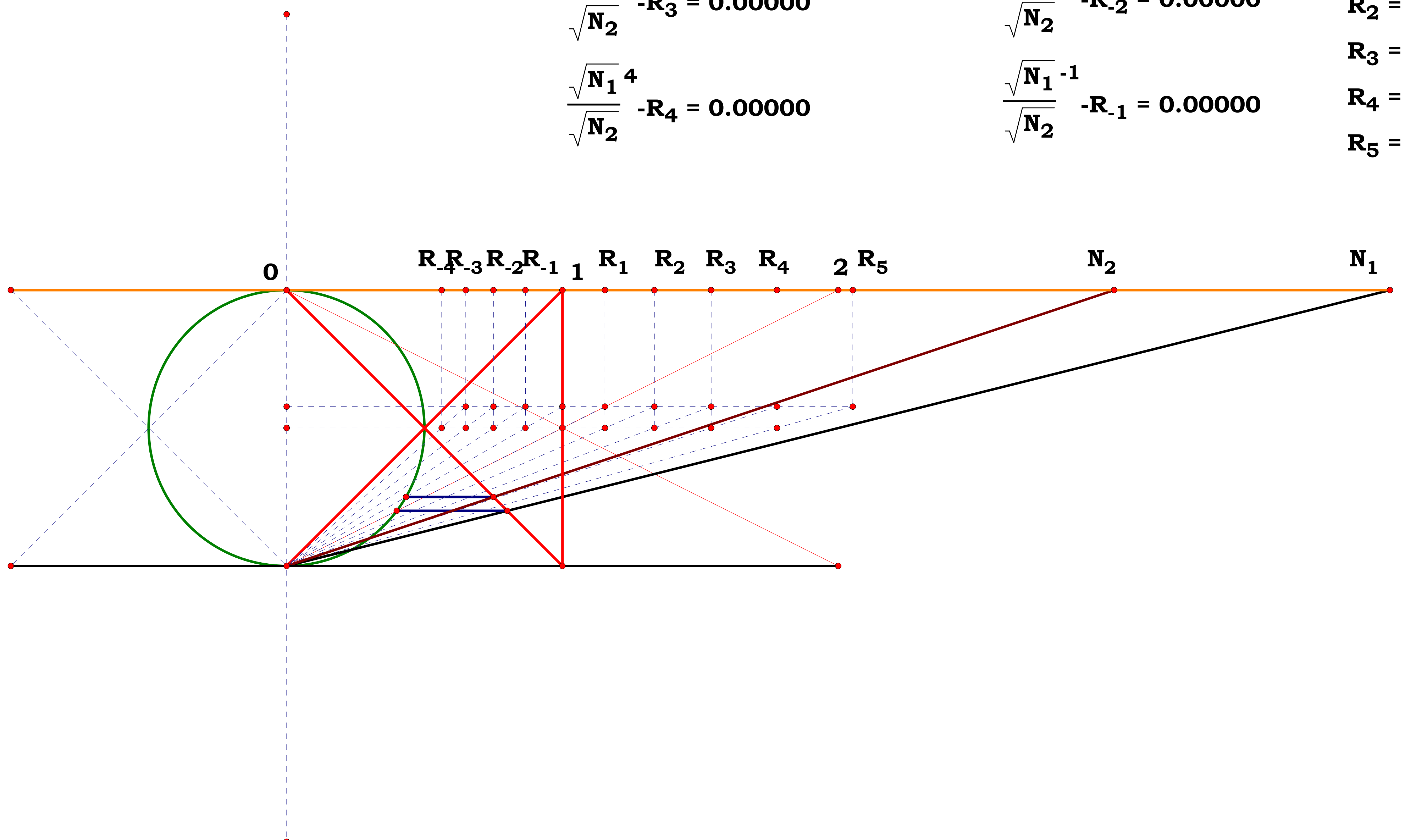
$R_4 = 1.77778$

$R_5 = 2.05280$

$N_1 = 4.00000$

$N_2 = 3.00000$

$$\frac{\sqrt{N_1}}{\sqrt{N_2}} = 1.15470$$



$$\frac{N_1}{N_1^{0.75}} = 1.15679$$

$$N_1 \cdot \left(\frac{N_1}{N_1^{0.75}}\right)^{-9} - R_{-9} = 0.00000$$

$$N_1 \cdot \left(\frac{N_1}{N_1^{0.75}}\right)^{-8} - R_{-8} = 0.00000$$

$$N_1 \cdot \left(\frac{N_1}{N_1^{0.75}}\right)^{-7} - R_{-7} = 0.00000$$

$$N_1 \cdot \left(\frac{N_1}{N_1^{0.75}}\right)^{-1} = 1.54798$$

$$N_1 \cdot \left(\frac{N_1}{N_1^{0.75}}\right)^{0} = 1.79070$$

$R_{-9} = 0.48275$

$R_{-8} = 0.55844$

$R_{-7} = 0.64600$

$R_{-6} = 0.74729$

$R_{-5} = 0.86446$

$R_{-3} = 1.15679$

$R_2 = 1.33817$

$R_3 = 1.54798$

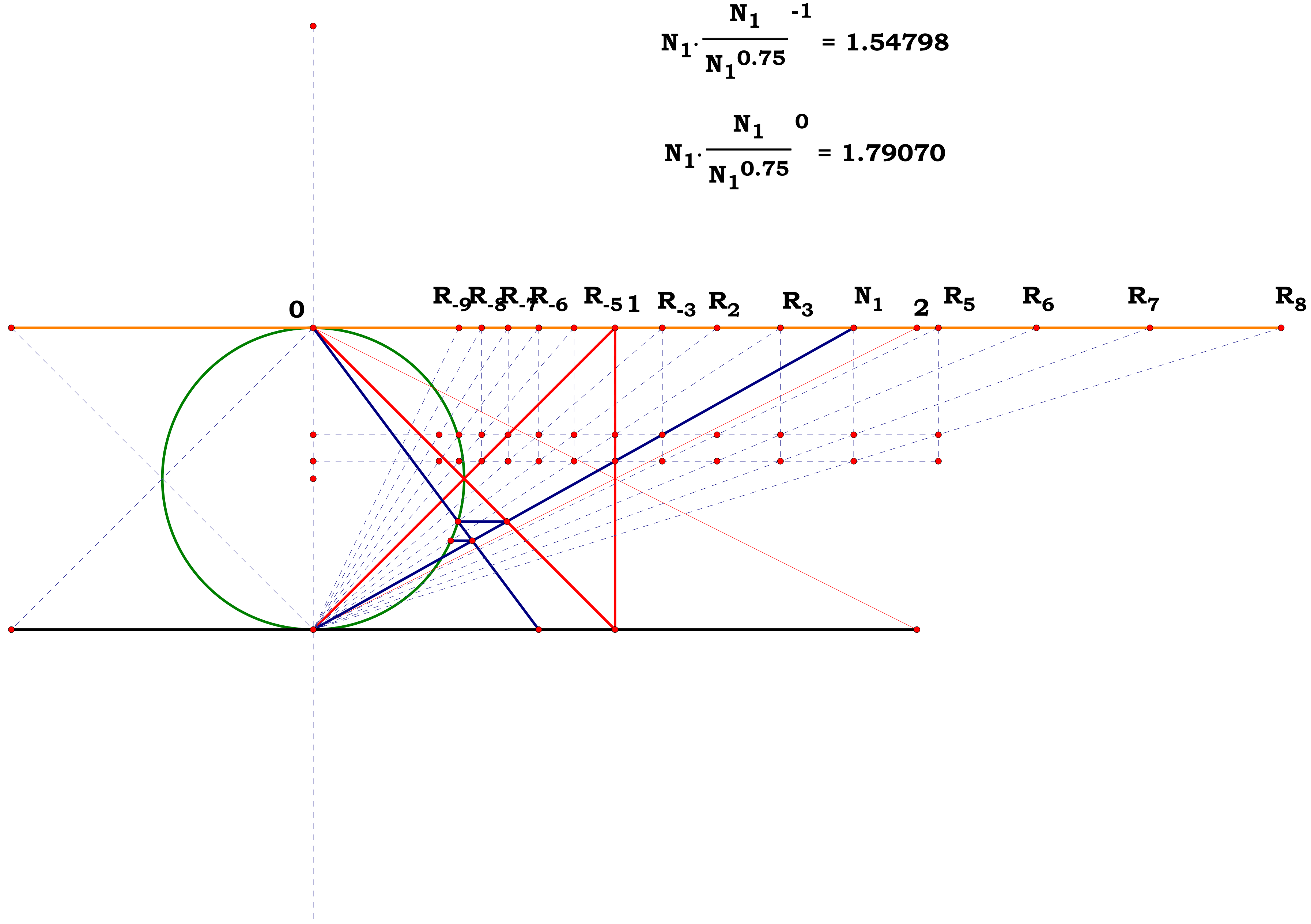
$N_1 = 1.79070$

$R_5 = 2.07147$

$R_6 = 2.39626$

$R_7 = 2.77197$

$R_8 = 3.20660$



Unit. $AB := 1$

Given. $C := .62081$

$X := .30123$

Descriptions.

$BC := 1 - C \quad BJ := BC \quad AG := \dfrac{AB}{BJ}$

$AD := 2 \cdot AB \quad DX := AD - X \quad FX := \sqrt{X \cdot DX} \quad GL := FX$

$BG := AG - AB \quad BX := \sqrt{(X - AB)^2}$

$Y := GL \cdot \dfrac{BX}{BG} \quad YO := 2 \cdot Y$

Definitions.

$BC - (1 - C) = 0 \quad BJ - (1 - C) = 0 \quad AG - \dfrac{1}{1 - C} = 0$

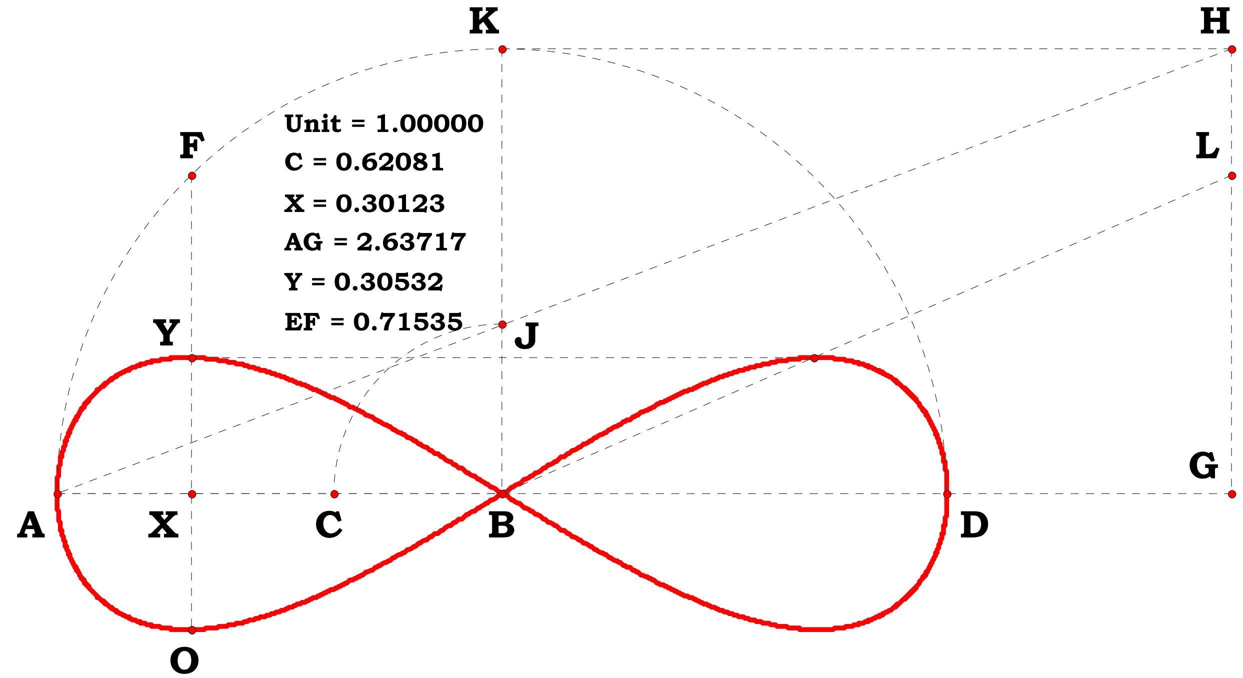
$AD - 2 = 0 \quad DX - (2 - X) = 0 \quad FX - \sqrt{X \cdot (2 - X)} = 0$

$GL - \sqrt{X \cdot (2 - X)} = 0 \quad BG - \dfrac{C}{1 - C} = 0$

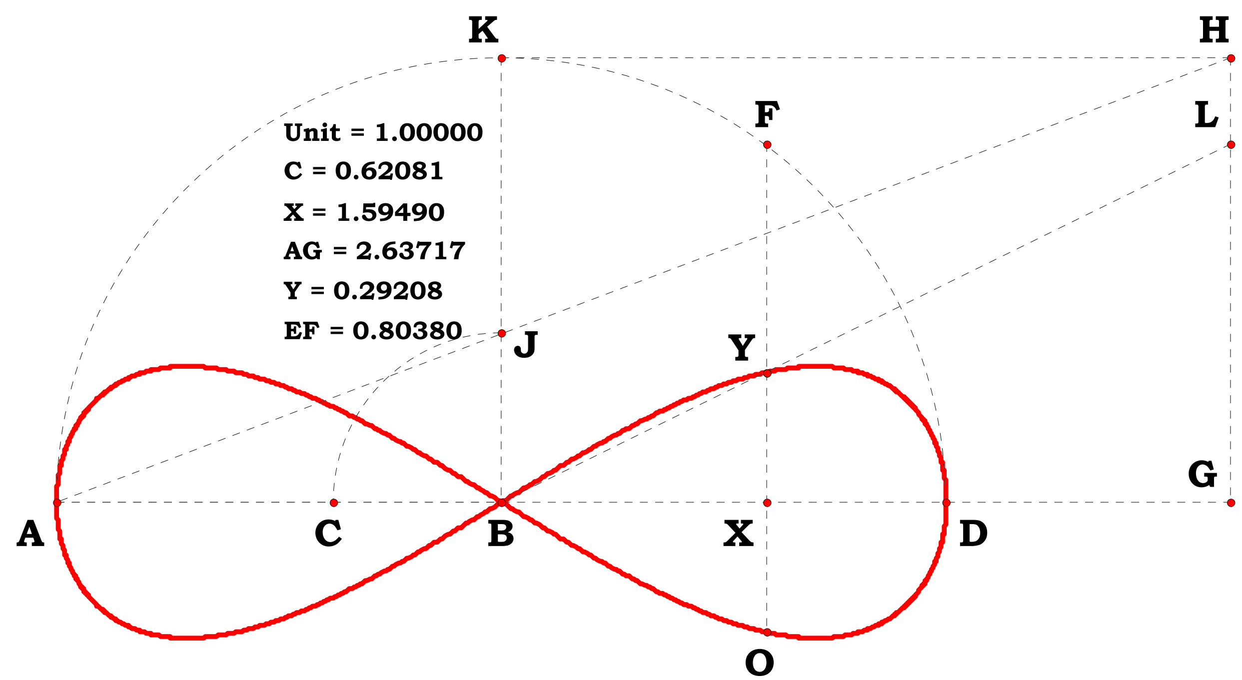
$BX - \sqrt{(X - 1)^2} = 0$

$Y - \dfrac{\sqrt{X \cdot (2 - X)} \cdot (1 - C) \cdot \sqrt{(X - 1)^2}}{C} = 0$

$YO - 2 \cdot \dfrac{\sqrt{X \cdot (2 - X)} \cdot (1 - C) \cdot \sqrt{(X - 1)^2}}{C} = 0$



$\dfrac{\sqrt{X \cdot (2-X)} \cdot (1-C) \cdot \sqrt{(X-1)^2}}{C} = 0.30532 \qquad \dfrac{\sqrt{X \cdot (2-X)} \cdot (1-C) \cdot \sqrt{(X-1)^2}}{C} - Y = 0.00000$



$\dfrac{\sqrt{X \cdot (2-X)} \cdot (1-C) \cdot \sqrt{(X-1)^2}}{C} = 0.29208 \qquad \dfrac{\sqrt{X \cdot (2-X)} \cdot (1-C) \cdot \sqrt{(X-1)^2}}{C} - Y = 0.00000$

Unit. $AB := 1$

Given. $C := .56747$

$X := .46656$

The Perfect Heart

Descriptions.

$AD := 2 \cdot AB \quad BC := AB - C \quad DX := AD - X$

$Y := \sqrt{X \cdot DX} \quad MN := 2 \cdot Y \quad BX := AB - X$

$BF := BX \cdot BC \quad FX := \sqrt{(BX - BF)^2} \quad M := Y + FX$

$N := FX - Y$

Definitions.

$AD - 2 = 0 \quad BC - (1 - C) = 0 \quad DX - (2 - X) = 0$

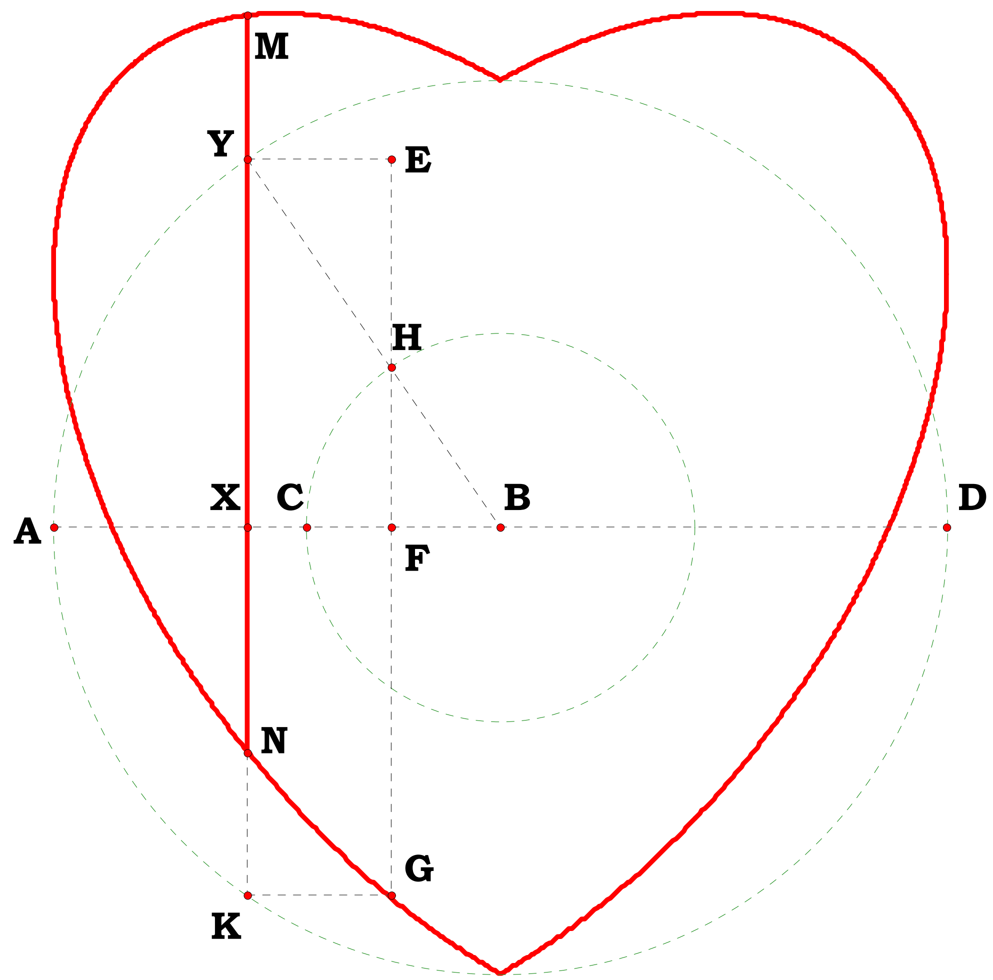
$Y - \sqrt{X \cdot (2 - X)} = 0 \quad MN - 2 \cdot \sqrt{X \cdot (2 - X)} = 0$

$BX - (1 - X) = 0 \quad BF - (1 - X) \cdot (1 - C) = 0$

$FX - \sqrt{C^2 \cdot (X - 1)^2} = 0$

$M - \left[Y + \sqrt{C^2 \cdot (X - 1)^2}\right] = 0$

$N - \left[\sqrt{C^2 \cdot (X - 1)^2} - Y\right] = 0$

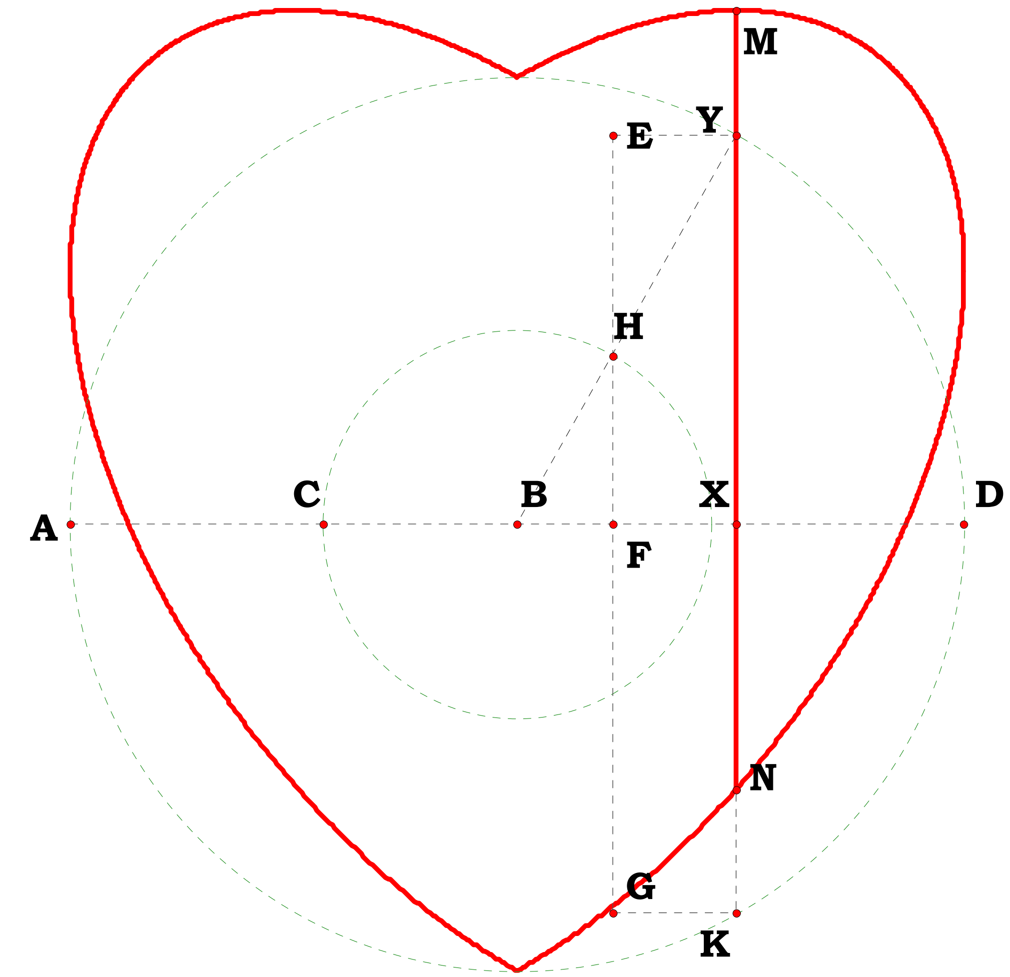


Unit = 1.00000
C = 0.56747
X = 0.43433
Y = 0.82463
M = 1.14563
N = -0.50364

$Y - \sqrt{X \cdot (2-X)} = 0.00000$

$M - \left(Y + \sqrt{C^2 \cdot (X-1)^2}\right) = 0.00000$

$N - \sqrt{C^2 \cdot (X-1)^2} - Y = 0.00000$



Unit = 1.00000
C = 0.56747
X = 1.49076
Y = 0.87130
M = 1.14978
N = -0.59281

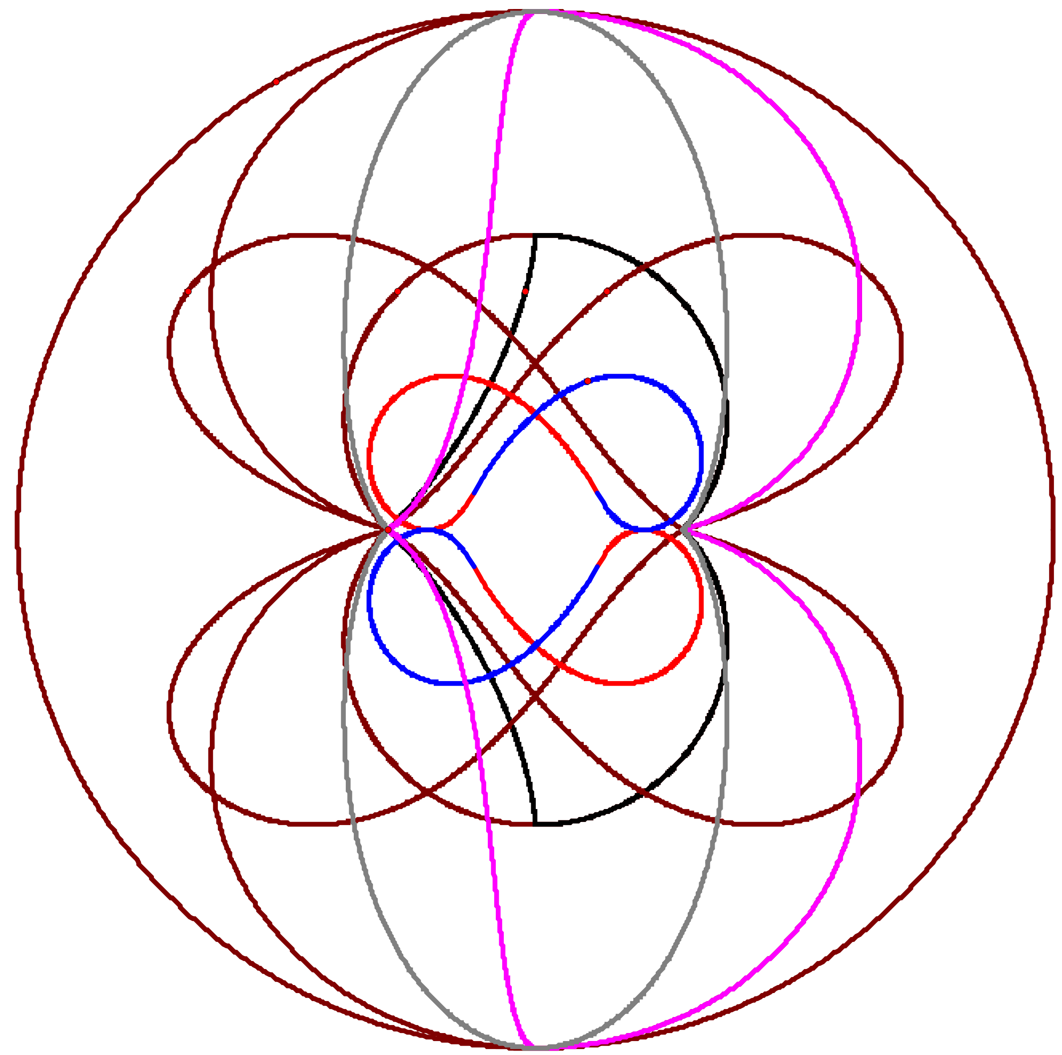
$Y - \sqrt{X \cdot (2-X)} = 0.00000$

$M - \left(Y + \sqrt{C^2 \cdot (X-1)^2}\right) = 0.00000$

$N - \sqrt{C^2 \cdot (X-1)^2} - Y = 0.00000$

The Rose, Bud

Friday, November 19, 2021



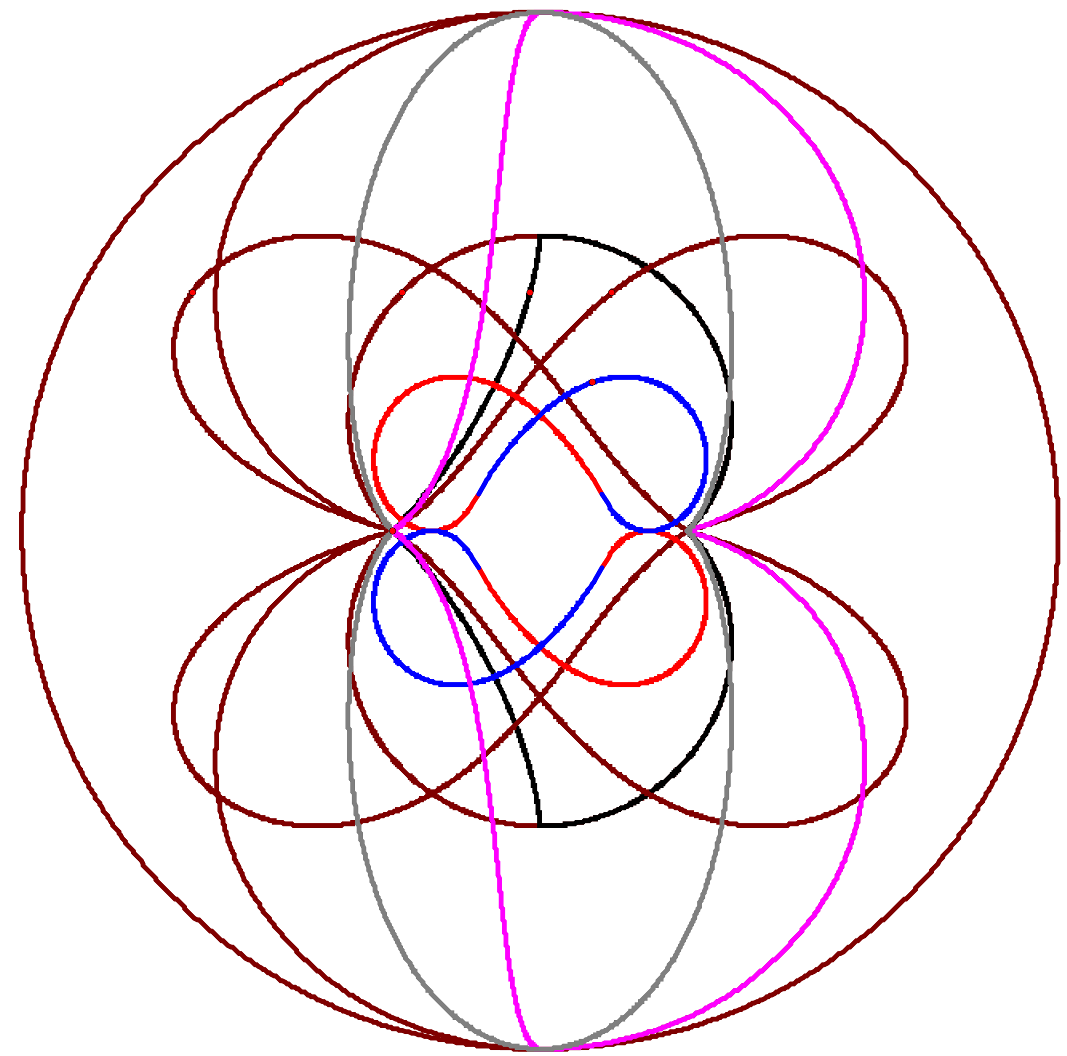
Is a series of loci all drawn on the same base. I will separate and write each of them up, and then have the resulting equation draw them. So, if you are wondering why some claim that there are higher plane curves which algebra cannot treat of, ask them this question: If Algebra is the third point in our grammar matrix, capable of complete induction and deduction by simple binary recursion, then do not your words imply that you are simply illiterate, effectively saying that binary recursion is, itself, different from binary recursion? And, if they are still skeptical, refer them

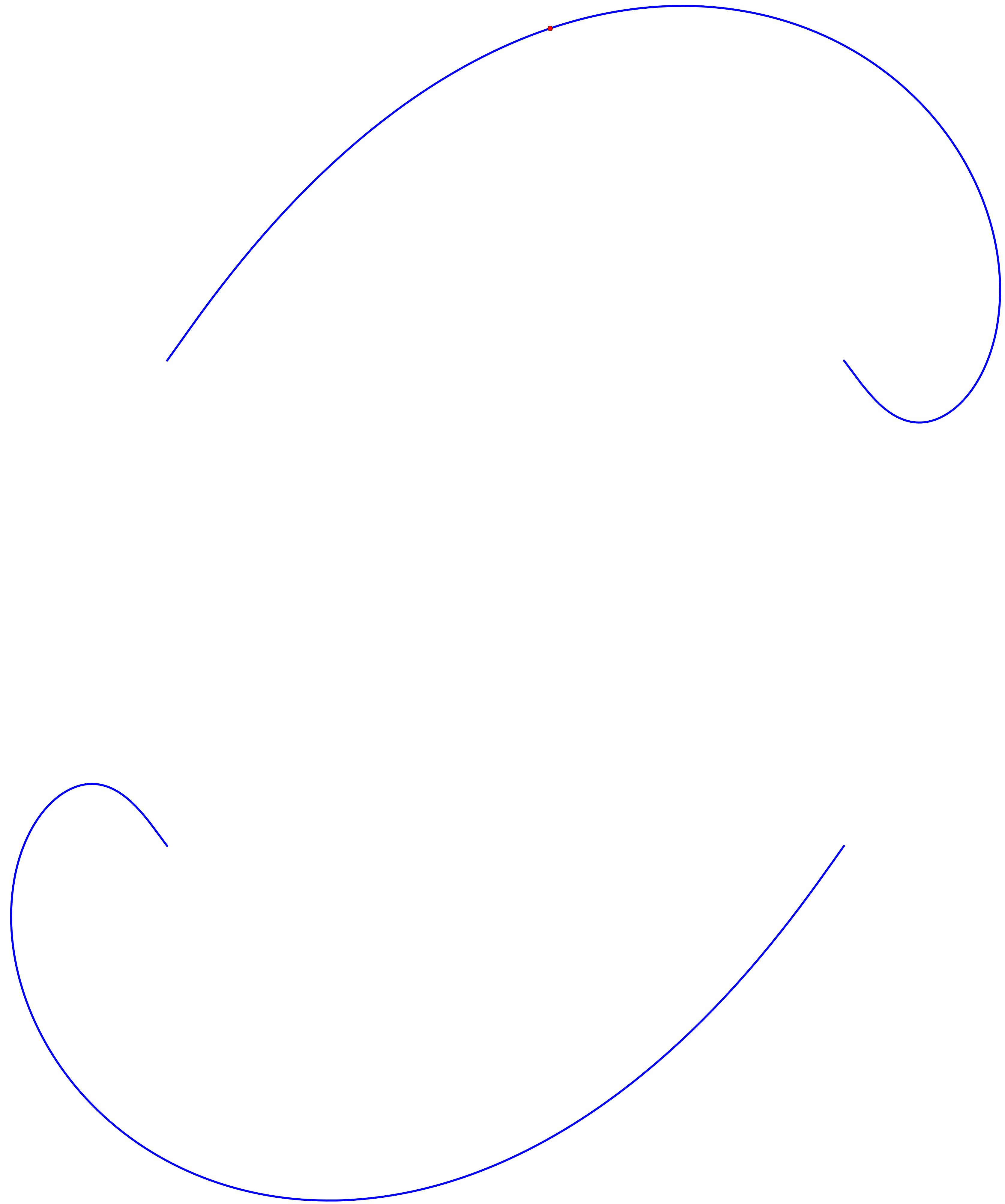
to the BAM Sample Dictionary where they can look up an infinite variety of curves, all drawn up using simple Algebra.

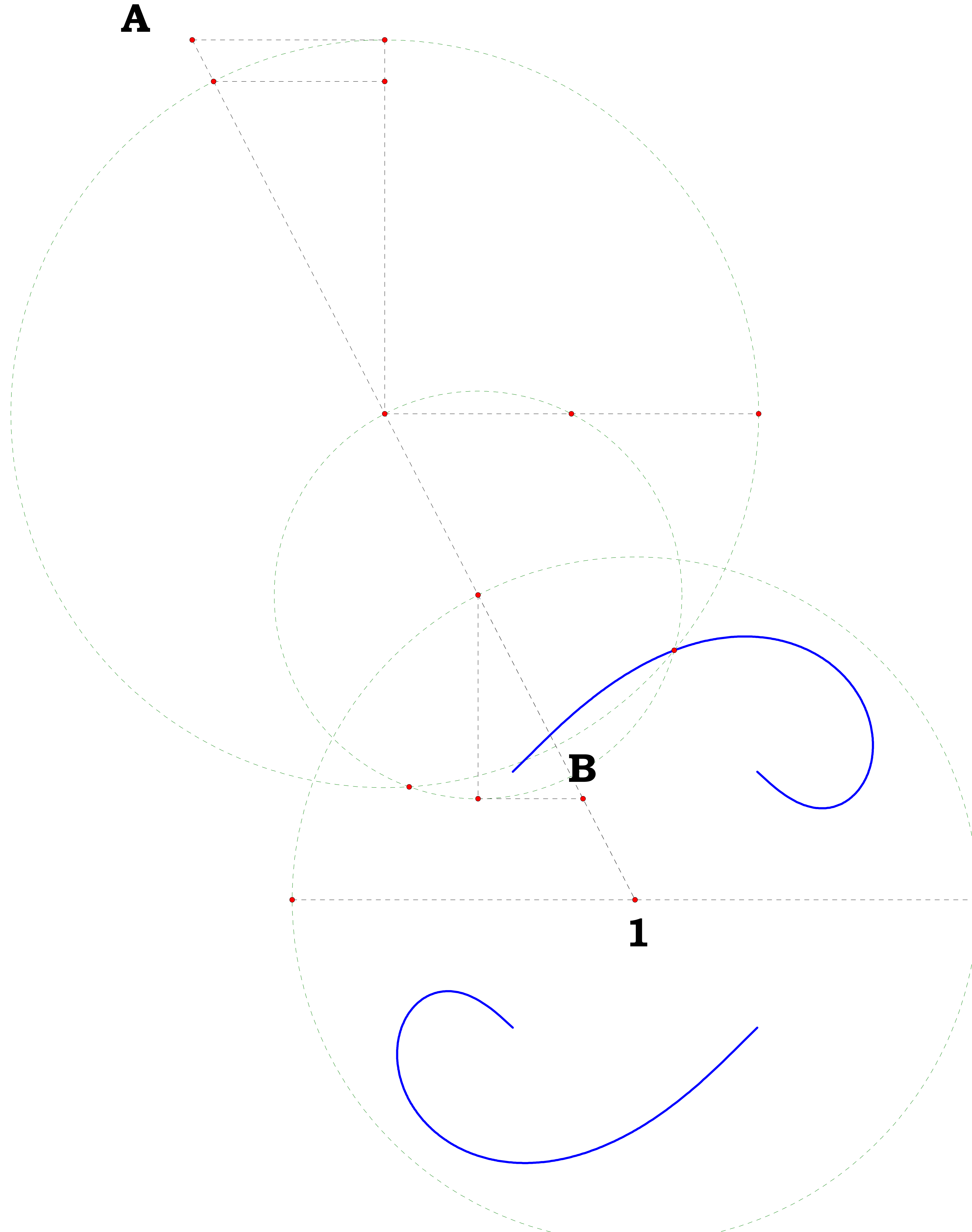
If I recall correctly, some of the curves one will find here, I once read in a book on calculous [sic], among all the gibberish about it, are simply impossible to demonstrate algebraically. I said to myself when reading it, Really? Okay, I will burn my work and buy a uniform. Every time I go to try and learn something about calculous [sic], I run into authors who speak noting but gibberish, not actually explaining anything without violating every known principle of grammar. So, I did a special on Newton's work, which, I might someday have time enough to study. However, I do not have much respect for authors who, unlike Plato, first realized the foundation of every possible system of grammar, meaning, that author did not have his mind tethered to factual reality from the start. For me, Calculus has always been a hard lump.

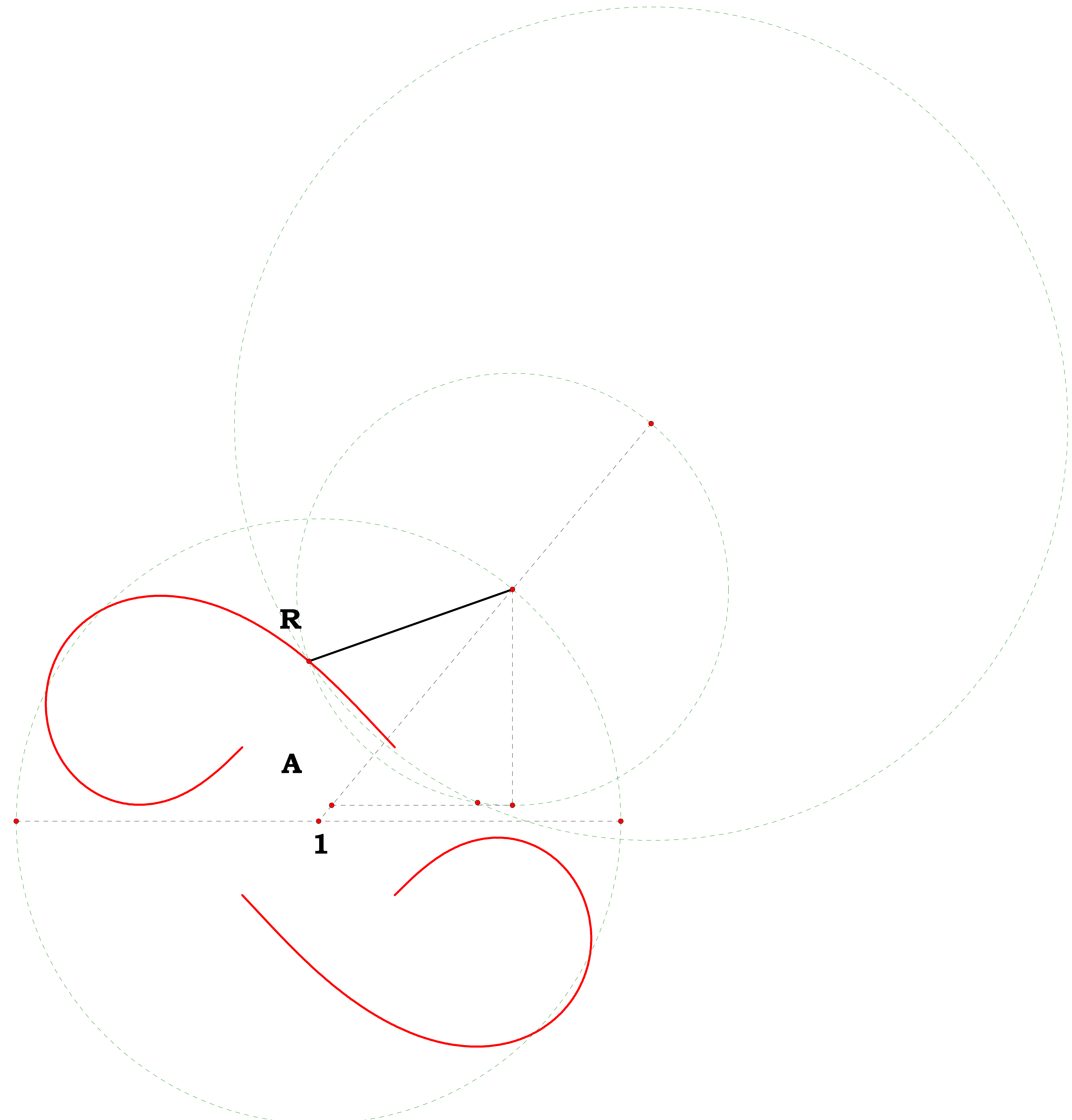
So, I went out to buy me a wall size poster of Cantor, who could not even comprehend the word complete, and call these curves, Transfinite Curves which only Calculus and be use to explain and forget that complete induction and deduction really means complete. I never found such a poster, and so I am waiting to be inducted into the Transfinite Reality of Spooky Action Association of Assentation Seekers, as soon as I can figure out how to dry my hair without a microwave oven.

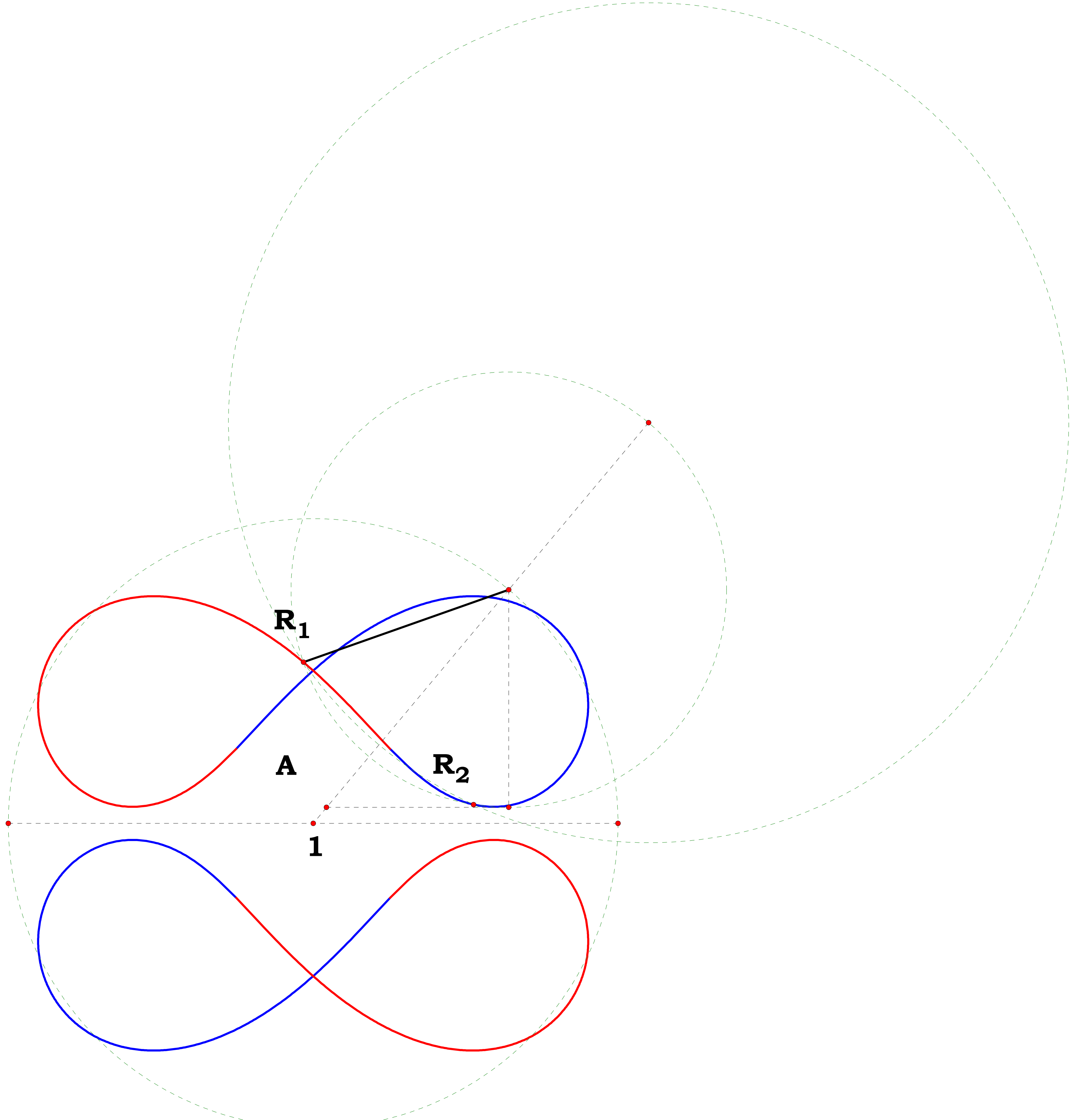
Only the basics required to draw the curve is written up, which means there is plenty one can do with finding the equations to the separate elements of those constructions.



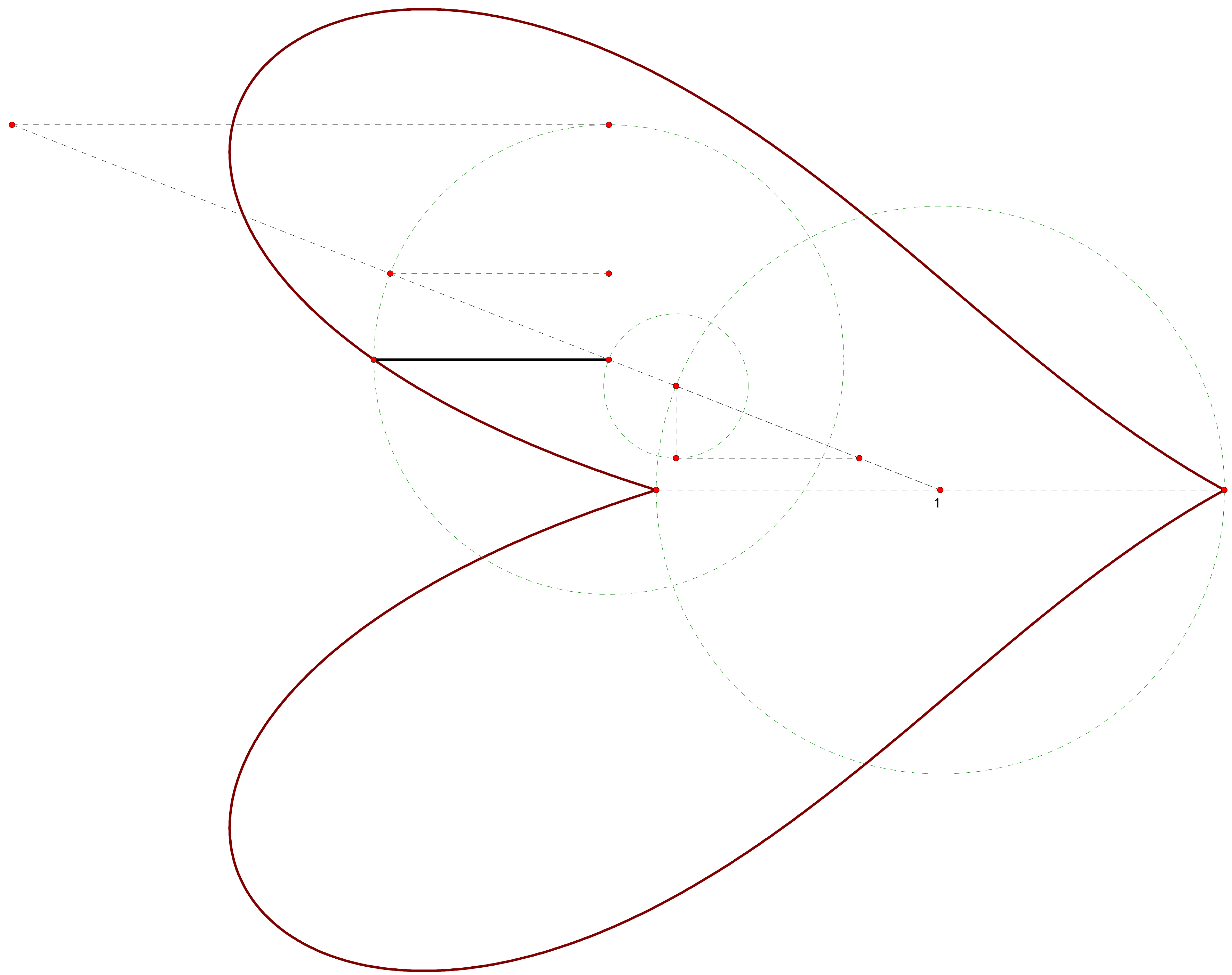



A
B
1


R
A
1



R1
A
R2
1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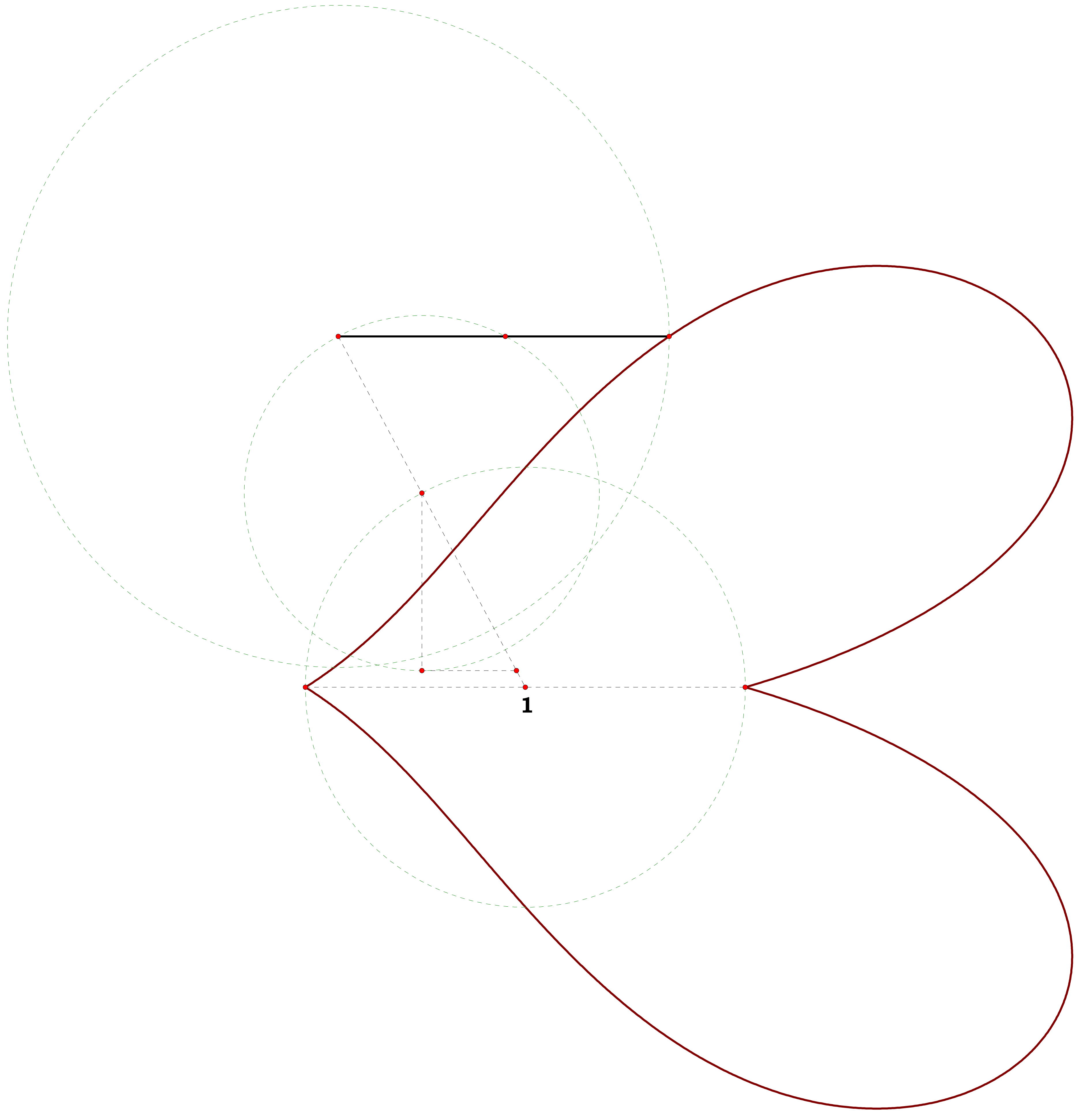
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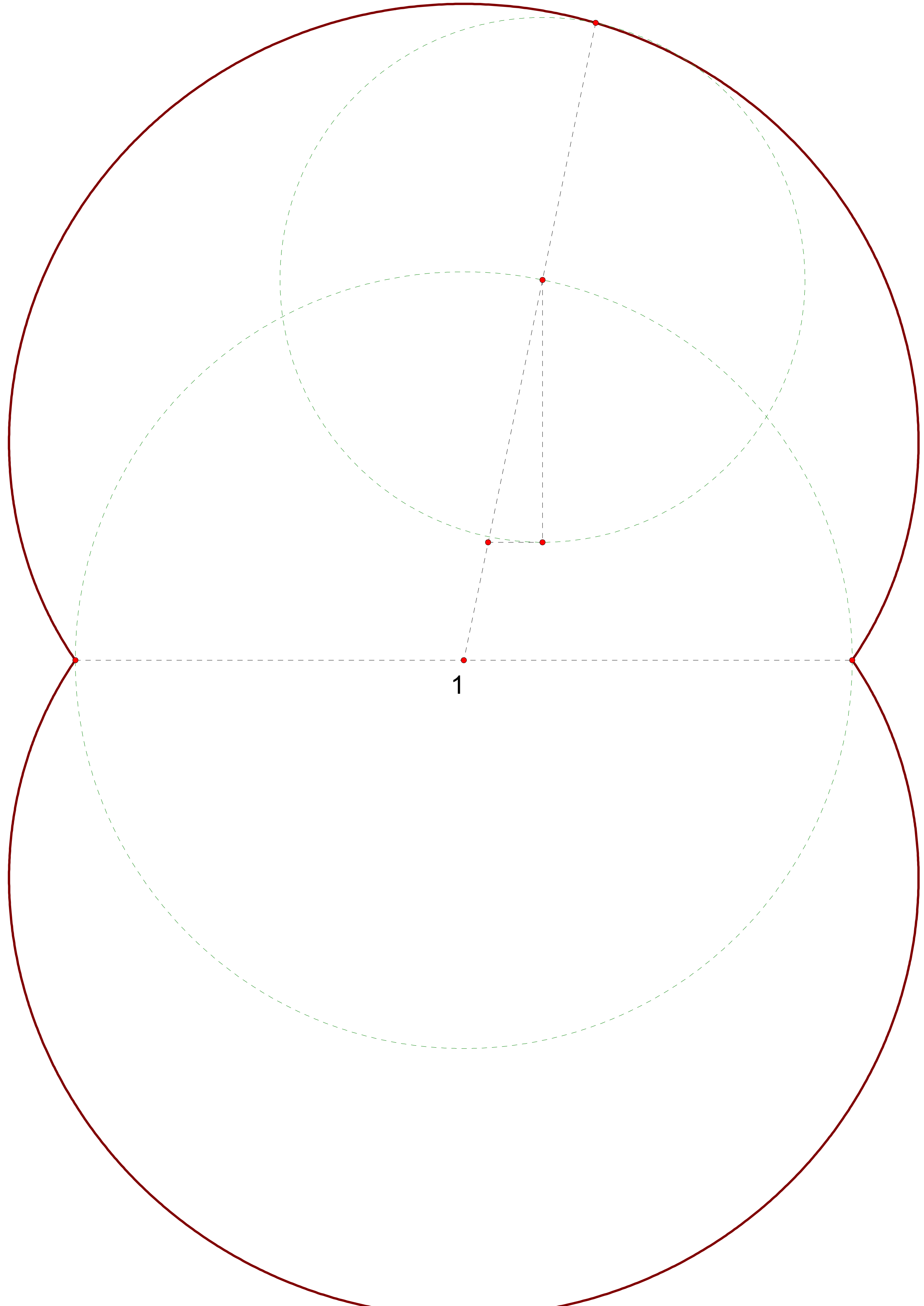
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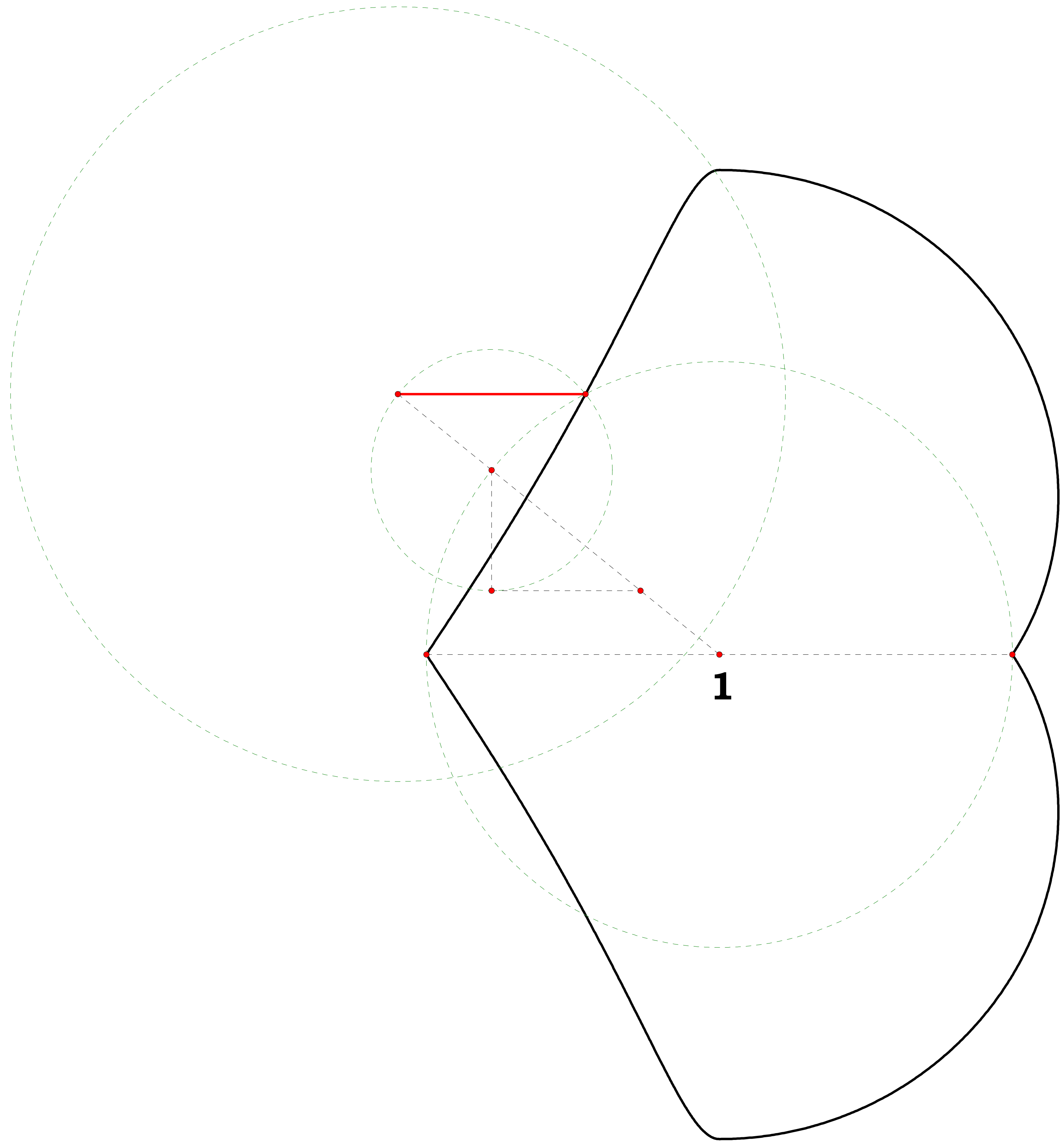
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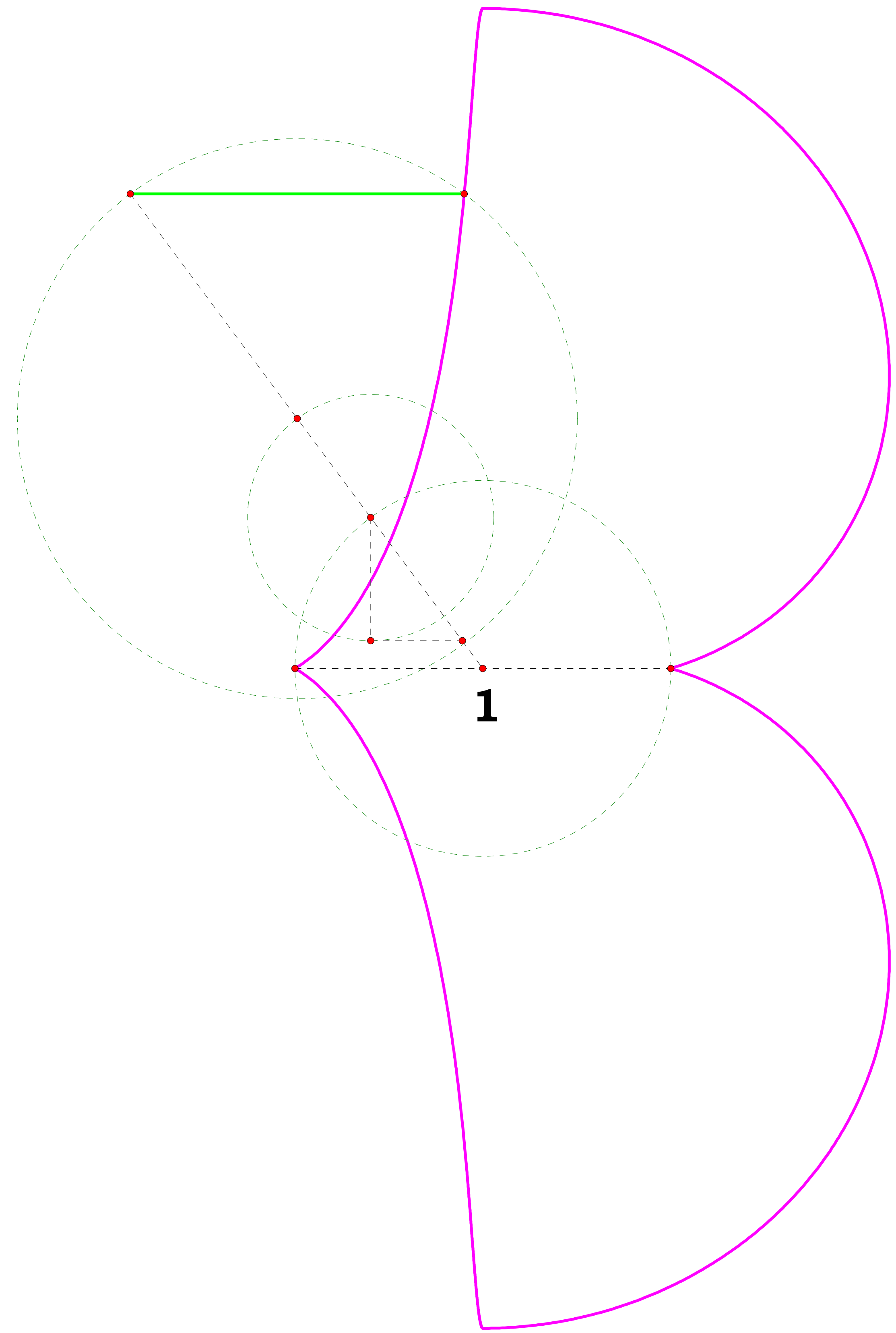
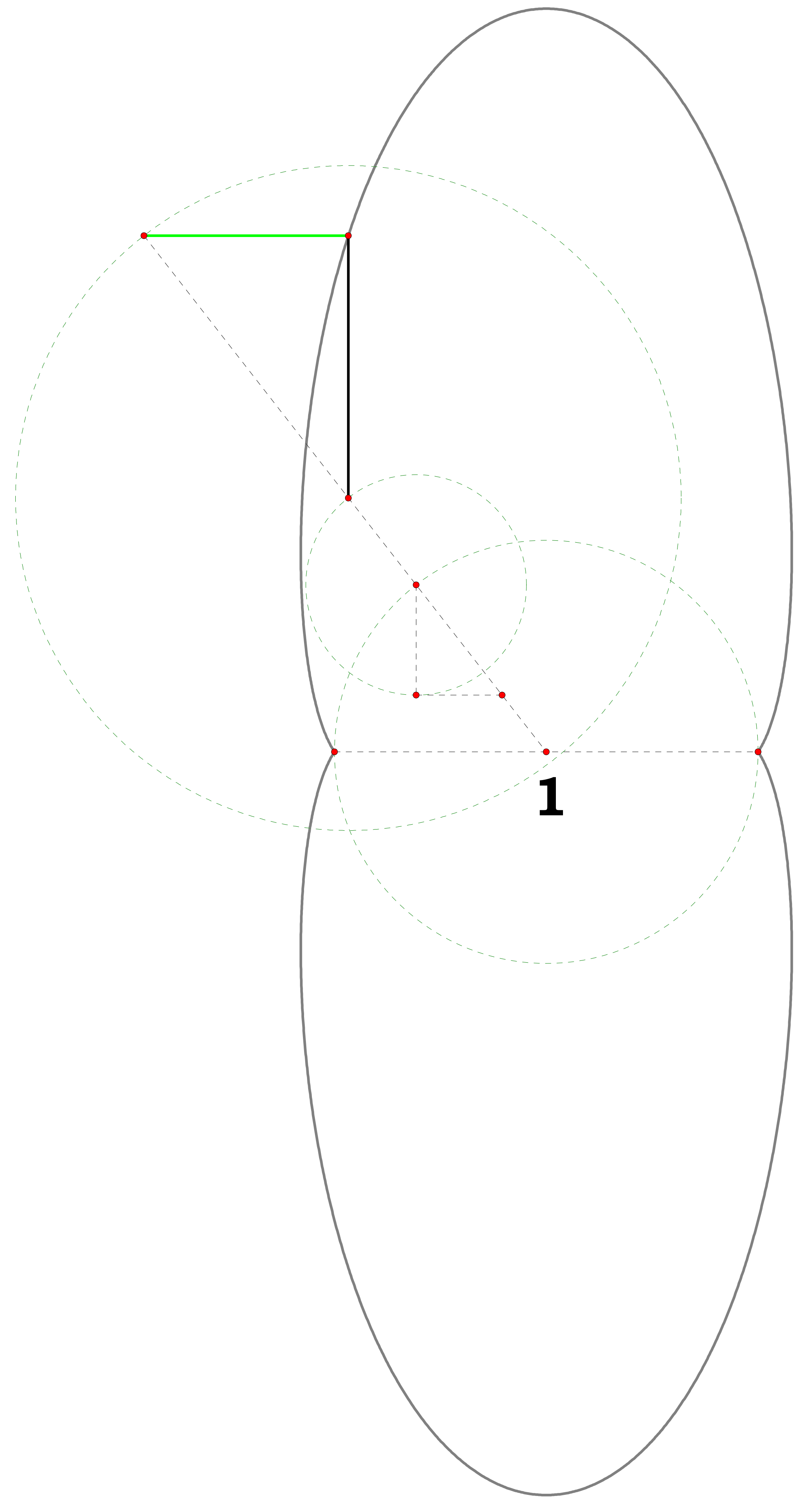
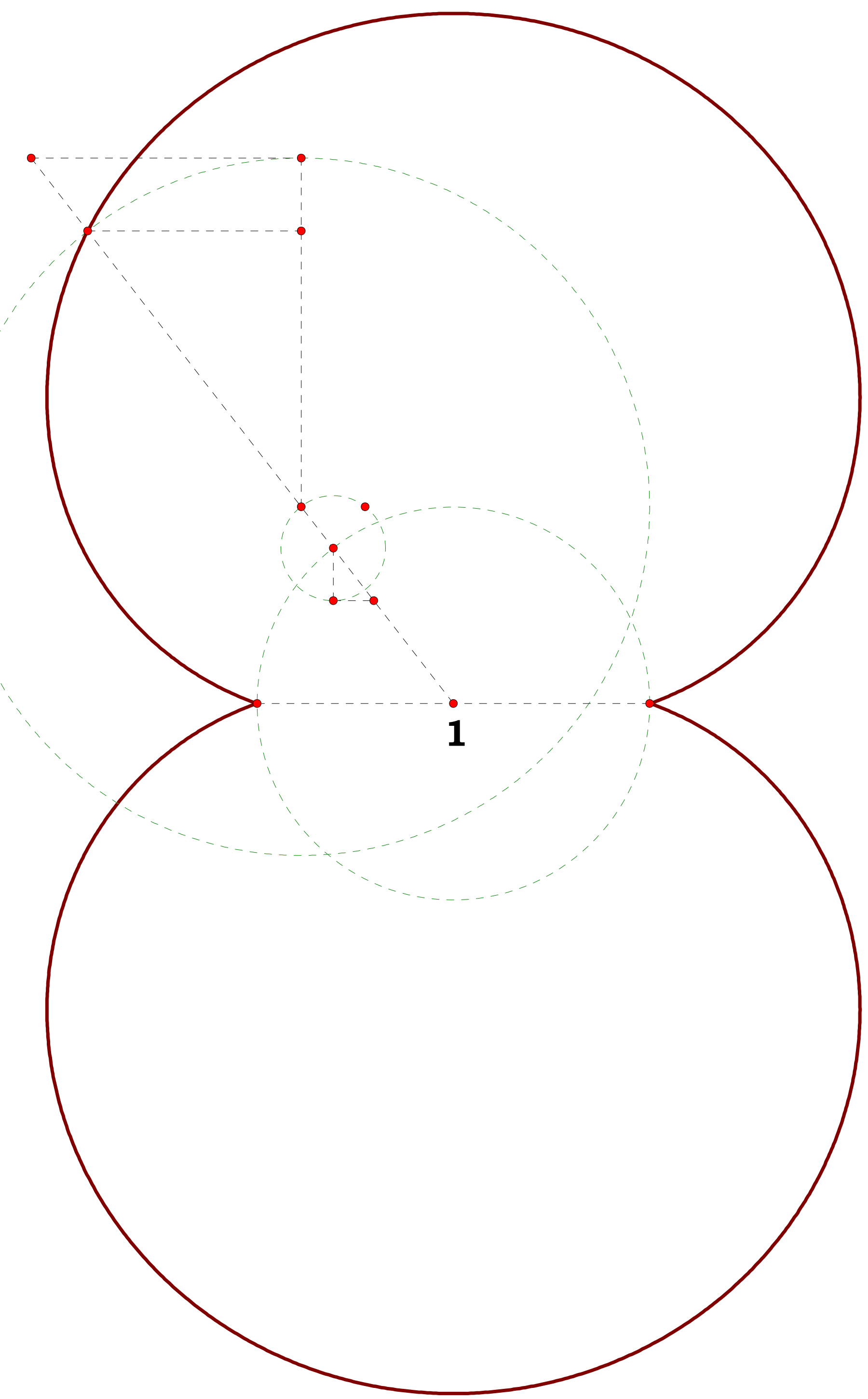


1



1


1

Unit. $AB := 1$

Given.

$C := .32838$

$X := 1.63143$

Tear Drop

Given the Licorice Yumyum, write the equation to a Hersey's Kiss

Descriptions.

$BC := 1 - C \quad BN := BC \quad AD := 2 \cdot AB \quad AM := \frac{AB^2}{BN}$

Licorice_Yumyum := BN

OM is the powerline between a Kiss and OMmmmm.

$DX := AD - X \quad FX := \sqrt{X \cdot DX} \quad GM := FX \quad Y := GM \cdot \frac{X}{AM}$

$JY := 2 \cdot Y$

Definitions.

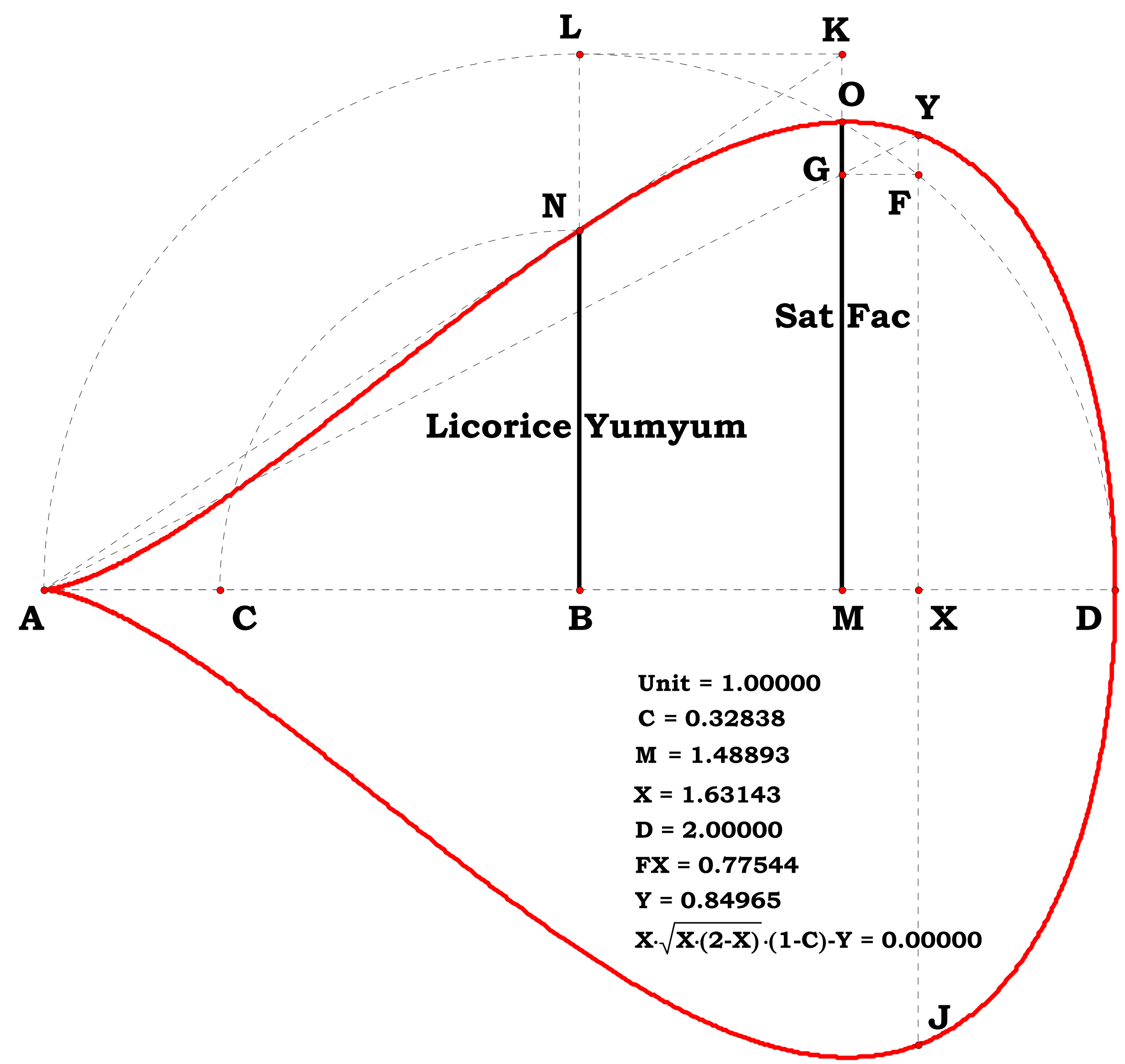
$BC - (1 - C) = 0 \quad BN - (1 - C) = 0 \quad AD - 2 = 0$

$AM - \frac{1}{1 - C} = 0 \quad DX - (2 - X) = 0$

$FX - \sqrt{[X \cdot (2 - X)]} = 0 \quad GM - \sqrt{[X \cdot (2 - X)]} = 0$

$Y - X \cdot \sqrt{X \cdot (2 - X)} \cdot (1 - C) = 0$

$JY - 2 \cdot X \cdot \sqrt{X \cdot (2 - X)} \cdot (1 - C) = 0$



Unit = 1.00000

C = 0.31365

X = 1.64157

a = 0.68635

1-C = 0.68635

b = 0.35843

2-X = 0.35843

X·(2-X) = 0.58839

d = 0.58839

$\sqrt{(X\cdot(2-X))}$ = 0.76706

e = 0.76706

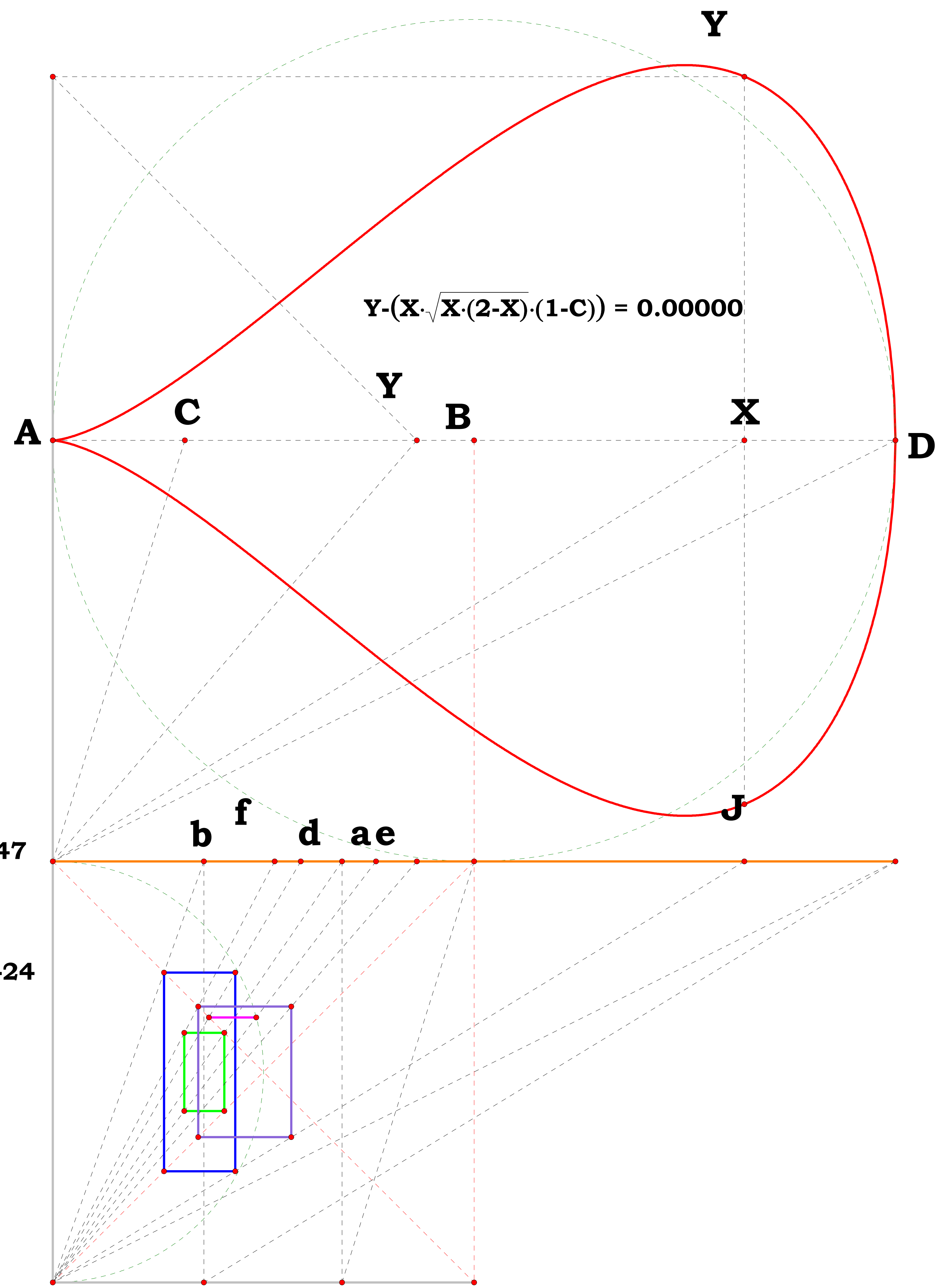
$\sqrt{(X\cdot(2-X))}\cdot(1-C)$ = 0.52647

f = 0.52647

X·f = 0.86424

$X\cdot\sqrt{X\cdot(2-X)}\cdot(1-C)$ = 0.86424

Y = 0.86424



The figure wrote the Algebra, the Algebra drew the figure.

Which came first? The chicken or the egg?
In an identity, which is first?

The X-Files

Wednesday, November 17, 2021

When I was a bit younger, I was reading an Algebra book, when, coming to the chapter on exponents, the author declared that exponential notation was a pure conceptual abstraction, that one could not example it in geometry. At that point, I knew my author was simply illiterate. Exponential manipulation, to anyone wielding a straightedge and compass should come as natural to anyone as it did me, after all, I am a grade-school, high-school, and college drop-out, i.e. a thrice confirmed drop-out and not in anywise a paper-trained professional mathematician who can show anyone how to scribble.

If one read up on doing powers and roots in geometry, some, for example cube roots are said to be impossible to demonstrate, which is a sad state of affairs: poor old man Apollo is still going door to door begging for an upgrade. Lucky for me, I had memories of a previous life-time to judge the situation.

In a previous incarnation, I was abducted by aliens; green ones, blue ones, not really any gray ones and some that were transparent like jelly-fish. During my Ordeals, I happened to see a copy of the *Elements* by Euclid written in some kind of alien lingo, so I took a peek, and lo and behold, I learnt the secret of exponential development with a straightedge and compass. It seems our alien neighbors mistook the criteria of straightedge and compass to mean something intelligible, like Plato. As all grammar systems are effected by simple binary recursion, the aliens took the segment as an absolute, and the compass a means of recursively applying it and thus, they could not distinguish, like our keen eyed geometers, the difference between a loci and a loci. Imagine that, aliens that cannot distinguish between binary recursion and binary recursion. I think they need to study the Babylon 5 episode where Zatharus explains

the difference between his many brothers named Zatharus and his name Zatharus.

I also learnt that Rene Descartes himself was almost abducted, but since he had his head up in the clouds, he would not fit into their space-craft. But, you can imagine how absurd it is to believe that Plato, who named the two-tools for geometry, to mean anything intelligible, nevertheless, it seems to work! I wonder to this day, if our alien neighbors left the book out just so I would take a peek at it. I do not, therefore, take any credit for those plates. I just copied them from memory.

So, this is a notebook of how I became mentally disturbed because of my alien abduction in a previous incarnation. I think in that life time, I was a woodpecker, which is very close to being a pencil pusher.

Any Root P1

Wednesday, November 17, 2021

Unit = 1.00000

A = 0.19203

B = 0.34719

C = 0.47255

D = 0.57383

$\frac{1}{1-A} = 1.23767$

$\frac{1}{1-B} = 1.53183$

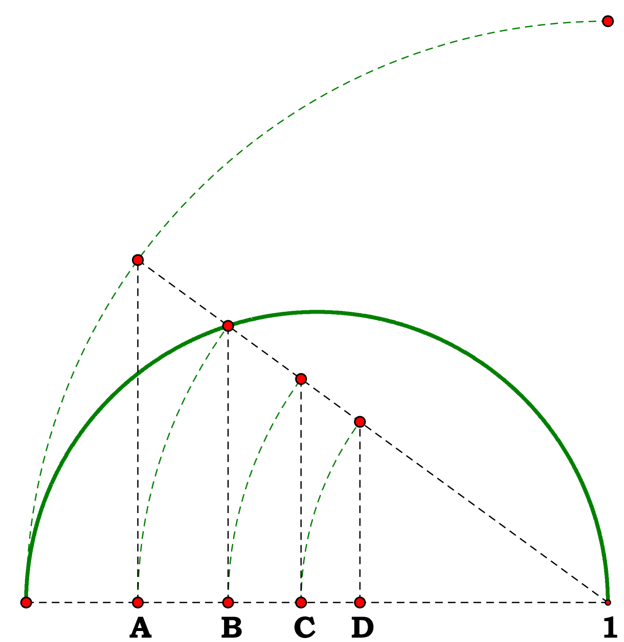
$\frac{1}{1-C} = 1.89590$

$\frac{1}{1-D} = 2.34651$

$\left(\frac{1}{1-D}\right)^{\frac{1}{4}} - \frac{1}{1-A} = 0.00000$

$\left(\frac{1}{1-D}\right)^{\frac{2}{4}} - \frac{1}{1-B} = 0.00000$

$\left(\frac{1}{1-D}\right)^{\frac{3}{4}} - \frac{1}{1-C} = 0.00000$



Many people have tried to find root series using Inductive reasoning, and conclude, well, it is not possible because of how quickly results extend out of sight: the extension is exponential which one's paper does not readily agree with. However, if one switch to deduction, the whole picture changes. In the early days of one playing with their pencil, they may try this and find root series very easily and the circle provides a ready environment for it, after all, many people have discovered the root, the so-called Pythagorean Theorem; the loci of the square root, indeed, just forms a smaller circle from the main circle. Therefore, N^1 and N^2, reside on a common curve. But that is as far as it goes, however, every root in that series is on exactly the same line, i.e. just another ratio. Given any adjacent two, in a series, one can then find the entire series for a given value. We take a hint from that the root projects from loci and simply fill in the missing loci and we have our root or power series. After all, a loci is a loci. If we listen to the figure, it will lead us forward.

Any Root P2

Wednesday, November 17, 2021

Now many people have stubbornly used up their whole life, looking for a way to get conics to perform this feat, but the curve, as one can plainly see, is simply not a conic.

Unit = 1.00000

A = 0.19203

B = 0.34719

C = 0.47255

D = 0.57383

$$\frac{1}{1-A} = 1.23767$$

$$\frac{1}{1-B} = 1.53183$$

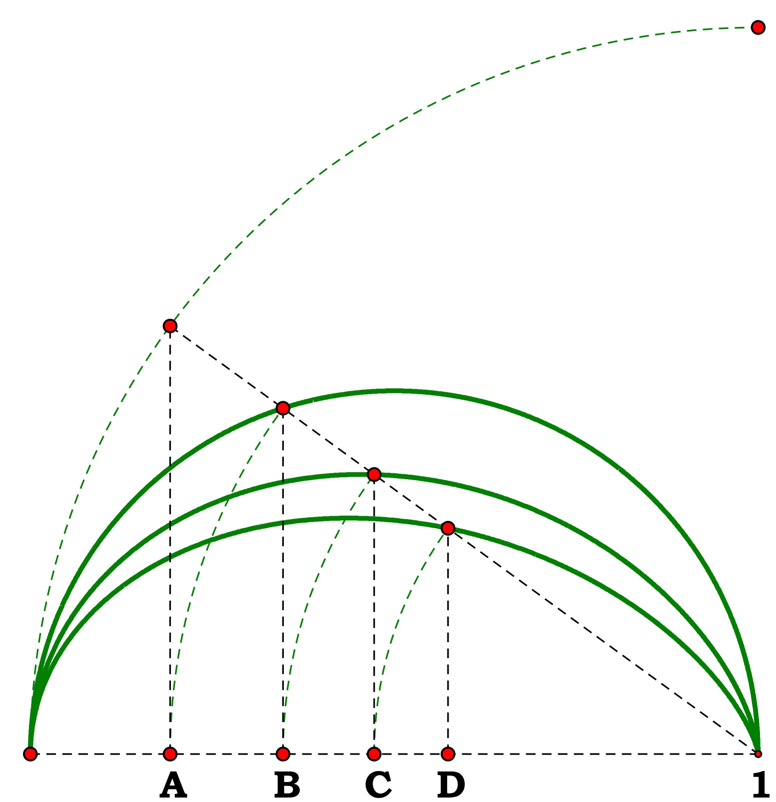
$$\frac{1}{1-C} = 1.89590$$

$$\frac{1}{1-D} = 2.34651$$

$$\left(\frac{1}{1-D}\right)^{\frac{1}{4}} - \frac{1}{1-A} = 0.00000$$

$$\left(\frac{1}{1-D}\right)^{\frac{2}{4}} - \frac{1}{1-B} = 0.00000$$

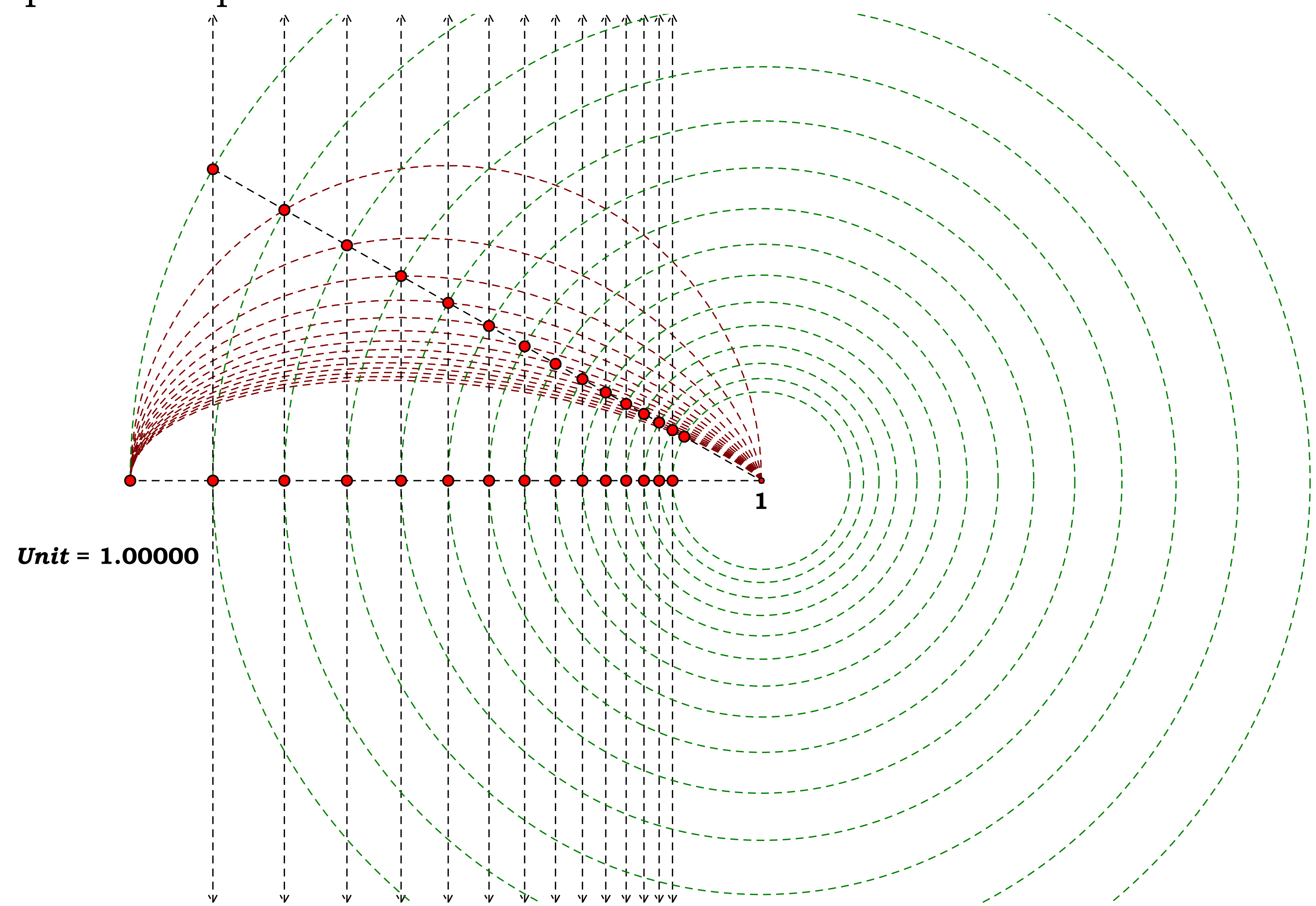
$$\left(\frac{1}{1-D}\right)^{\frac{3}{4}} - \frac{1}{1-C} = 0.00000$$



Any Root P3

Wednesday, November 17, 2021

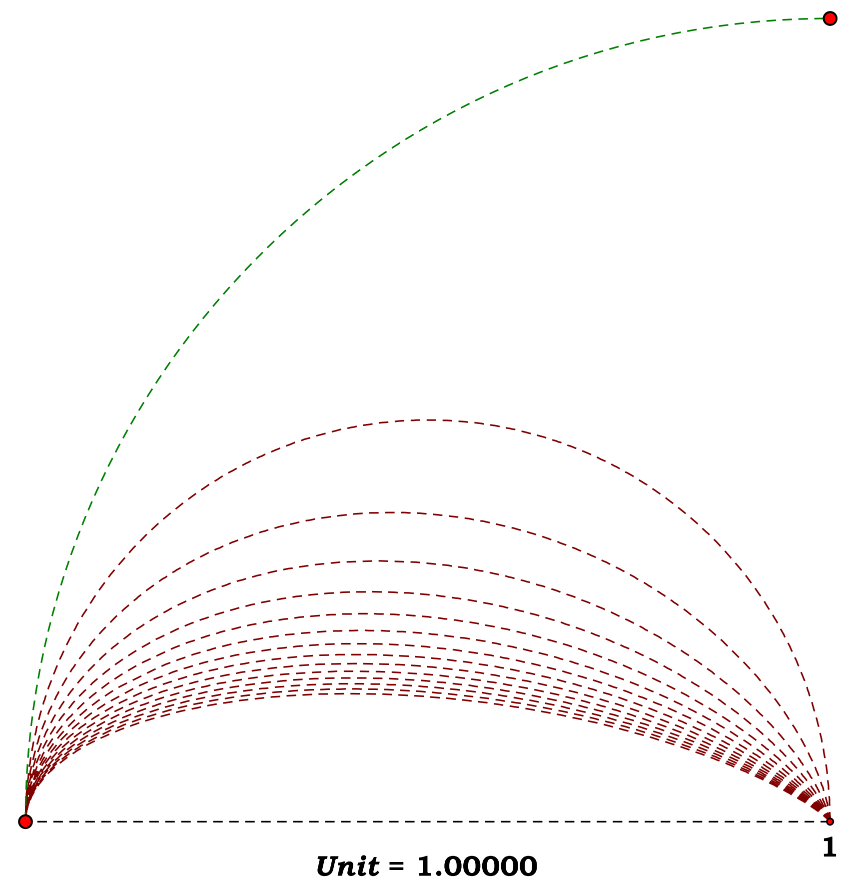
I was hired by this hypnotist to construct spiffy patterns to help put people to sleep.



Any Root P4.

Wednesday, November 17, 2021

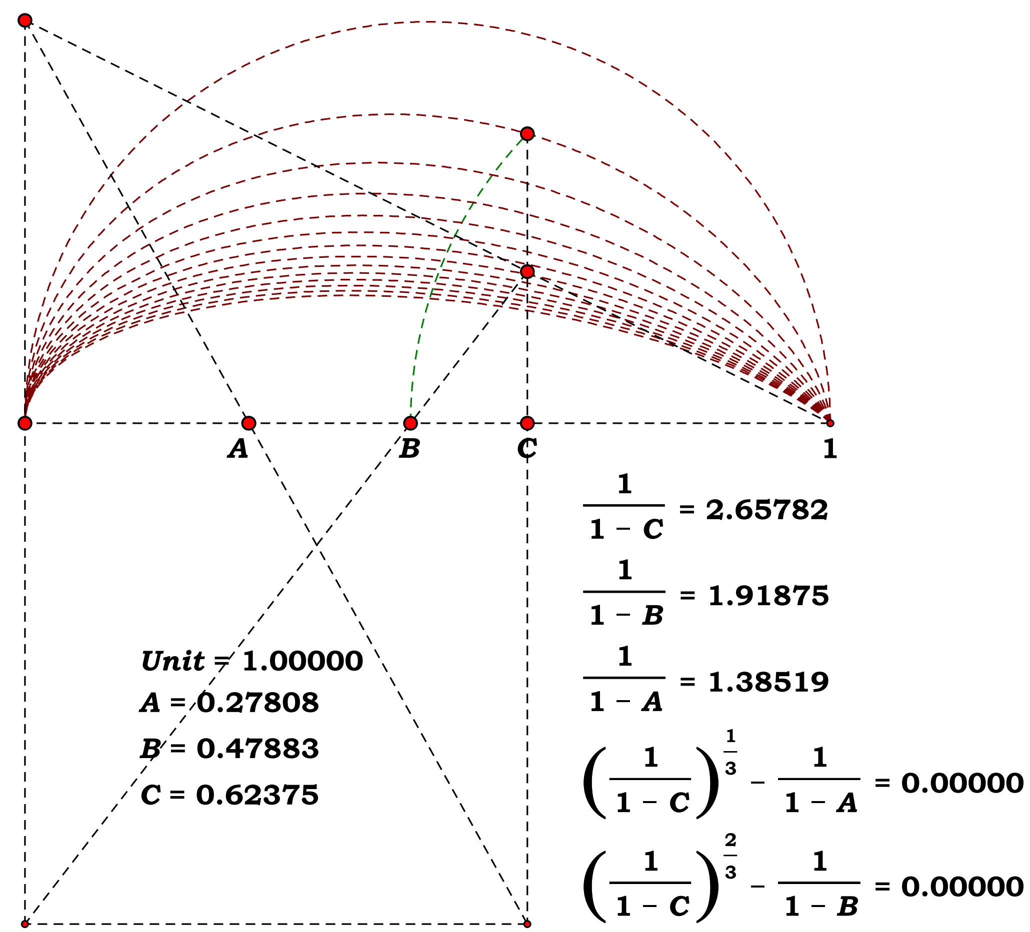
With just a few more iterations, one will have discovered the perfect airfoil.

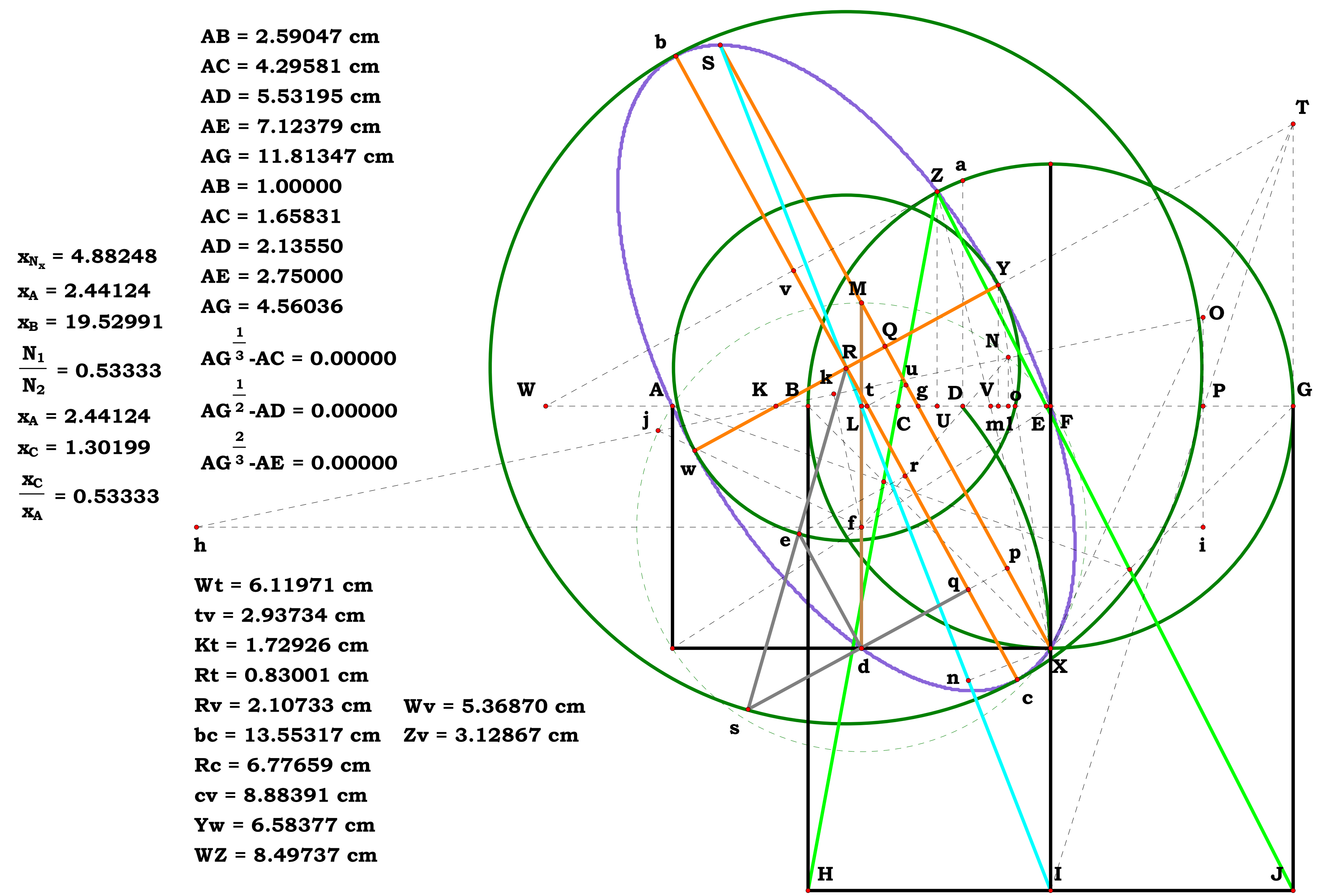


Any Root P5

Wednesday, November 17, 2021

Look ma, no plague! On July 2nd 246 B. C. Apollo got a delivery from Amazon of his new Altar. I noticed, when I first heard about the Delian Problem, that in metaphor, it means exactly the same thing as the challenge in Genesis: Man must learn judgment, i.e. the principles of grammar. In order to duplicate the cube, then one has to learn the grammar of geometry as just another form of binary recursion. I also notice that people whose mind cannot resolve a metaphor, that they miss the message with the result that they produce a whole lot of mythology and crackpot science. It is wholly impossible for an illiterate mind to produce rational results. I was pleased however when I could solve it both ways, inductively and deductively. It does use a conic, but everything about that conic is given in the figure itself. All one has to do is draw it.

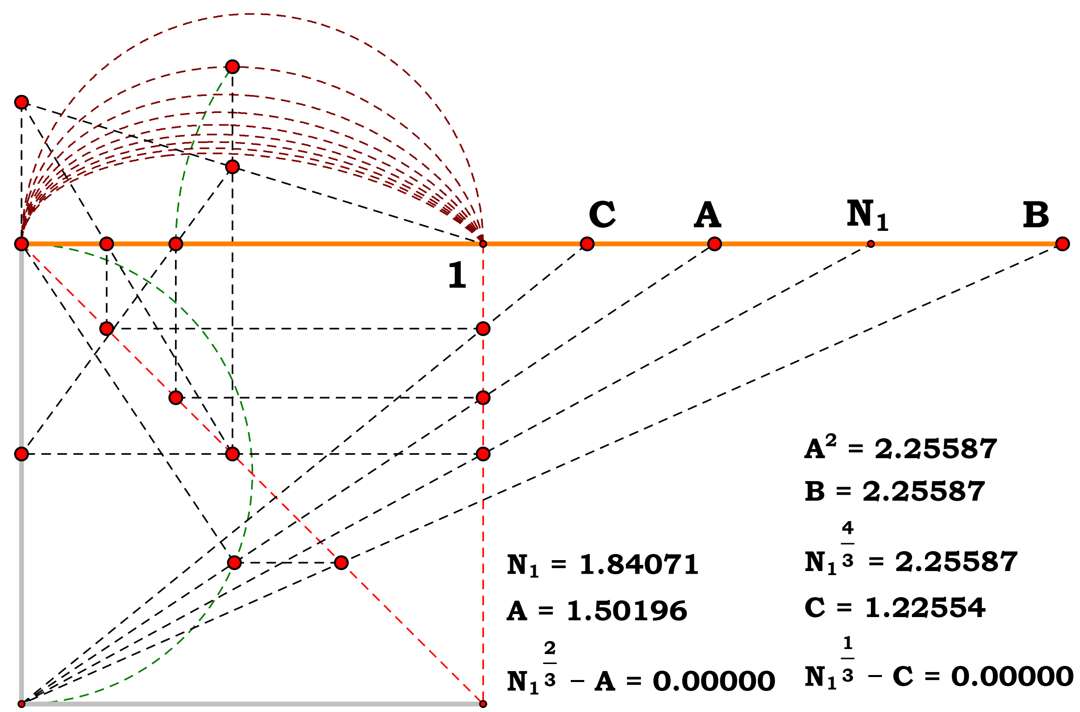




Granted, it is a whole lot more complicated, however, the satisfaction in being able to produce it was commensurable. And so this is one root which can be solved both inductively and deductively on the very same page of a piece of paper! (you just have to be prepared to go through the alphabet twice. The starting square root figure does lay down everything one needs to simply fill in the conic for cube roots. One will find the whole story in *The Delian Quest.*

Any Root P37

Thursday, November 18, 2021



$N_1 = 1.84071$

$A = 1.50196$

$N_1^{\frac{2}{3}} - A = 0.00000$

$A^2 = 2.25587$

$B = 2.25587$

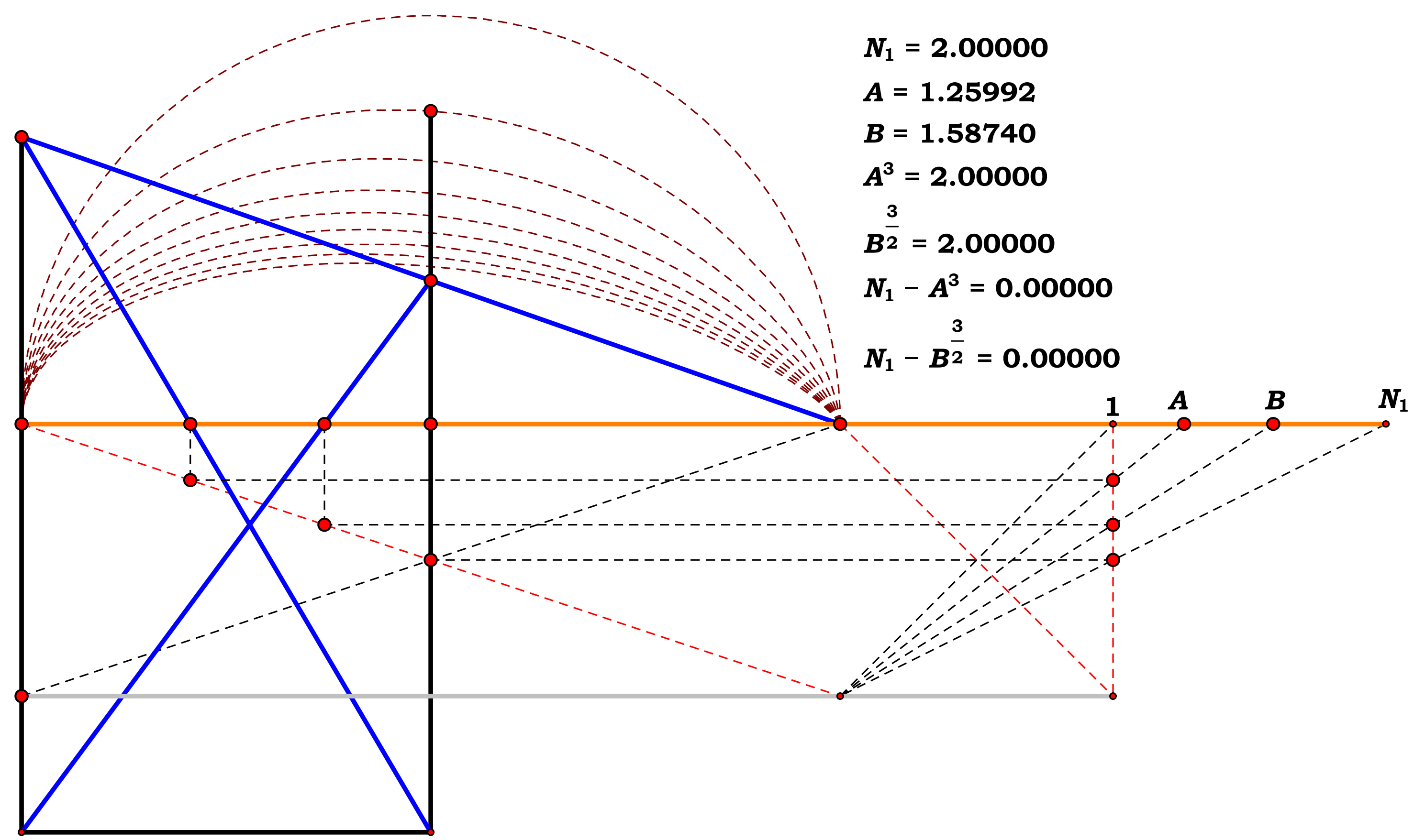
$N_1^{\frac{4}{3}} = 2.25587$

$C = 1.22554$

$N_1^{\frac{1}{3}} - C = 0.00000$

Any Root P38

Thursday, November 18, 2021



Any Root P39

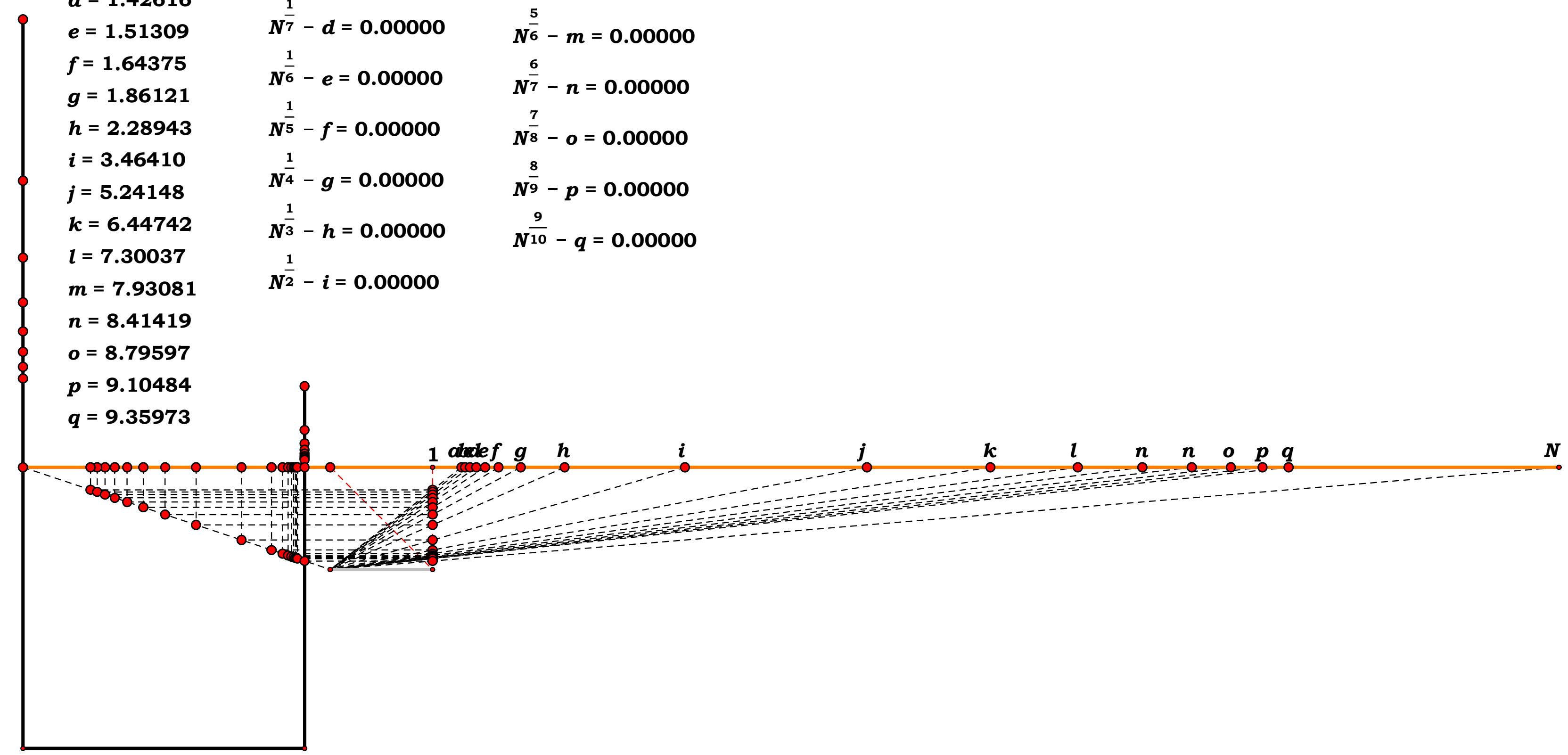
Thursday, November 18, 2021

$N = 12.00000$

$a = 1.28209$
$b = 1.31798$
$c = 1.36426$
$d = 1.42616$
$e = 1.51309$
$f = 1.64375$
$g = 1.86121$
$h = 2.28943$
$i = 3.46410$
$j = 5.24148$
$k = 6.44742$
$l = 7.30037$
$m = 7.93081$
$n = 8.41419$
$o = 8.79597$
$p = 9.10484$
$q = 9.35973$

$N^{\frac{1}{10}} - a = 0.00000$
$N^{\frac{1}{9}} - b = 0.00000$
$N^{\frac{1}{8}} - c = 0.00000$
$N^{\frac{1}{7}} - d = 0.00000$
$N^{\frac{1}{6}} - e = 0.00000$
$N^{\frac{1}{5}} - f = 0.00000$
$N^{\frac{1}{4}} - g = 0.00000$
$N^{\frac{1}{3}} - h = 0.00000$
$N^{\frac{1}{2}} - i = 0.00000$

$N^{\frac{2}{3}} - j = 0.00000$
$N^{\frac{3}{4}} - k = 0.00000$
$N^{\frac{4}{5}} - l = 0.00000$
$N^{\frac{5}{6}} - m = 0.00000$
$N^{\frac{6}{7}} - n = 0.00000$
$N^{\frac{7}{8}} - o = 0.00000$
$N^{\frac{8}{9}} - p = 0.00000$
$N^{\frac{9}{10}} - q = 0.00000$



Any Root 1-2

Wednesday, November 17, 2021

$Unit = 1.00000$

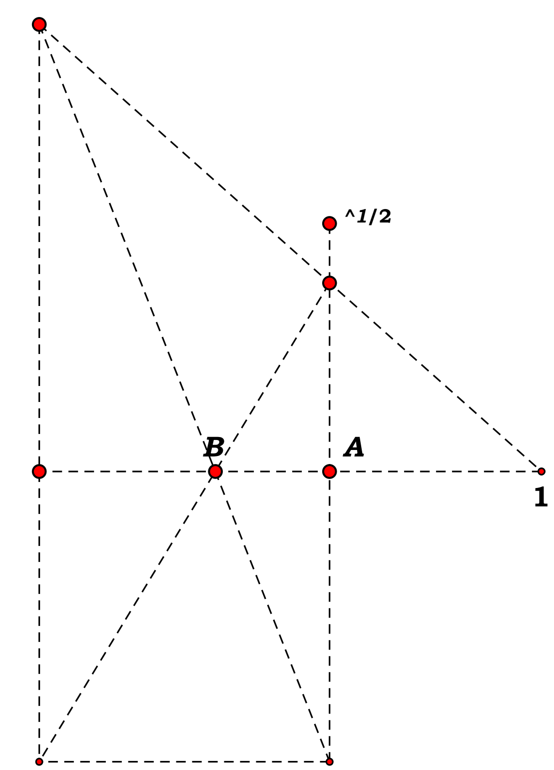
$A = 0.57838$

$B = 0.35068$

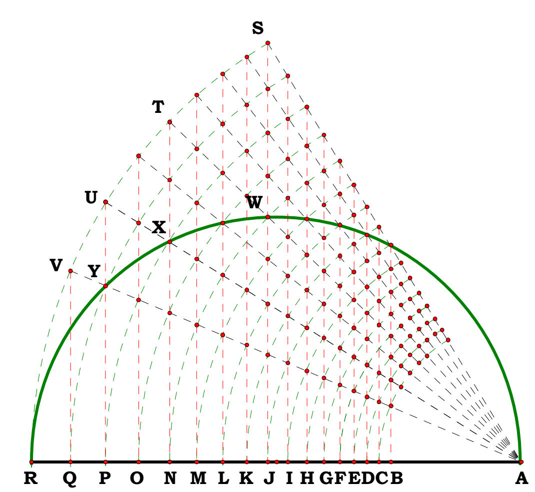
$$\frac{1}{1-A} = 2.37182$$

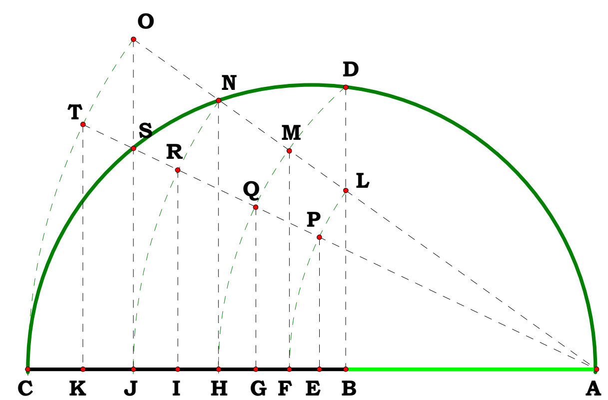
$$\frac{1}{1-B} = 1.54007$$

$$\left(\frac{1}{1-A}\right)^{\frac{1}{2}} - \frac{1}{1-B} = 0.00000$$



It might not look like much because it isn't. It is just a reminder, that the root series can be found with any pair of roots, which includes the unlabeled point and A, or the substance of one's starting value. Once one has them, as in the case of the square root, one can find every 2^n where n factors as a series of 2's,root series with it, i.e. 2, 4, 8, 16, 32, etc.





$AC^0 = 1.00000$	$N_1 = 1.00000$	$\frac{AB}{AB} = 1.00000$	AC = 3.95000 in.
$AC^{\frac{1}{8}} = 1.10784$	AE = 1.10784	$\frac{AE}{AB} = 1.10784$	AB = 1.74089 in.
$AC^{\frac{2}{8}} = 1.22732$	AF = 1.22732	$\frac{AF}{AB} = 1.22732$	AE = 1.92864 in.
$AC^{\frac{3}{8}} = 1.35967$	AG = 1.35967	$\frac{AG}{AB} = 1.35967$	AF = 2.13663 in.
$AC^{\frac{4}{8}} = 1.50630$	AH = 1.50630	$\frac{AH}{AB} = 1.50630$	AG = 2.36704 in.
$AC^{\frac{5}{8}} = 1.66875$	AI = 1.66875	$\frac{AI}{AB} = 1.66875$	AH = 2.62231 in.
$AC^{\frac{6}{8}} = 1.84871$	AJ = 1.84871	$\frac{AJ}{AB} = 1.84871$	AI = 2.90511 in.
$AC^{\frac{7}{8}} = 2.04808$	AK = 2.04808	$\frac{AK}{AB} = 2.04808$	AJ = 3.21841 in.
$AC^{\frac{8}{8}} = 2.26895$	AC = 2.26895	$\frac{AC}{AB} = 2.26895$	AK = 3.56549 in.

$AC^0 - N_1 = 0.00000$

$AC^{\frac{1}{8}} - AE = 0.00000$

$AC^{\frac{2}{8}} - AF = 0.00000$

$AC^{\frac{3}{8}} - AG = 0.00000$

$AC^{\frac{4}{8}} - AH = 0.00000$

$AC^{\frac{5}{8}} - AI = 0.00000$

$AC^{\frac{6}{8}} - AJ = 0.00000$

$AC^{\frac{7}{8}} - AK = 0.00000$

$AC^{\frac{8}{8}} - AC = 0.00000$

$N_1 = 2.97175$

$N_1^{\frac{8}{8}} = 2.97175$

$N_1^{\frac{7}{8}} = 2.59350$

$N_1^{\frac{6}{8}} = 2.26339$

$N_1^{\frac{5}{8}} = 1.97530$

$N_1^{\frac{4}{8}} = 1.72388$

$N_1^{\frac{3}{8}} = 1.50446$

$N_1^{\frac{2}{8}} = 1.31296$

$N_1^{\frac{1}{8}} = 1.14585$

$N_1^{\frac{9}{8}} = 3.40517$

$N_1^{\frac{10}{8}} = 3.90181$

$N_1^{\frac{11}{8}} = 4.47087$

$N_1^{\frac{-1}{8}} = 0.87272$

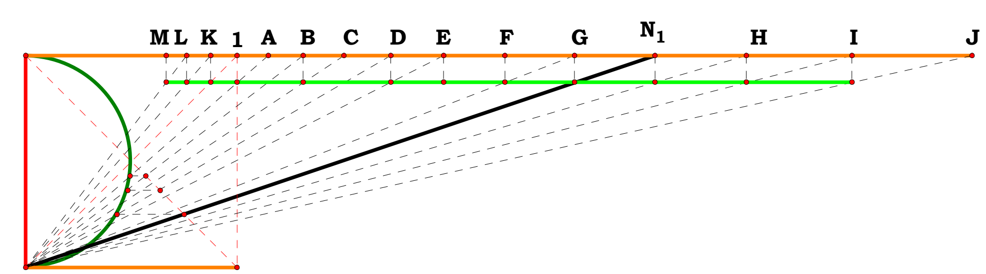
$N_1^{\frac{-2}{8}} = 0.76163$

$N_1^{\frac{-3}{8}} = 0.66469$

A = 1.14585
B = 1.31296
C = 1.50446
D = 1.72388
E = 1.97530
F = 2.26339
G = 2.59350
H = 3.40517
I = 3.90181
J = 4.47087
K = 0.87272
L = 0.76163
M = 0.66469

$N_1^0 = 1.00000$

$N_1^{\frac{8}{8}} - N_1 = 0.00000$



$N_1^{\frac{-1}{8}} - K = 0.00000$

$N_1^{\frac{-2}{8}} - L = 0.00000$

$N_1^{\frac{-3}{8}} - M = 0.00000$

$N_1^{\frac{1}{8}} - A = 0.00000$

$N_1^{\frac{2}{8}} - B = 0.00000$

$N_1^{\frac{3}{8}} - C = 0.00000$

$N_1^{\frac{4}{8}} - D = 0.00000$

$N_1^{\frac{5}{8}} - E = 0.00000$

$N_1^{\frac{6}{8}} - F = 0.00000$

$N_1^{\frac{7}{8}} - G = 0.00000$

$N_1^{\frac{9}{8}} - H = 0.00000$

$N_1^{\frac{10}{8}} - I = 0.00000$

$N_1^{\frac{11}{8}} - J = 0.00000$

Any Root 1-3

Wednesday, November 17, 2021

Unit **= 1.00000**

$A = 0.59888$

$B = 0.45610$

$C = 0.26250$

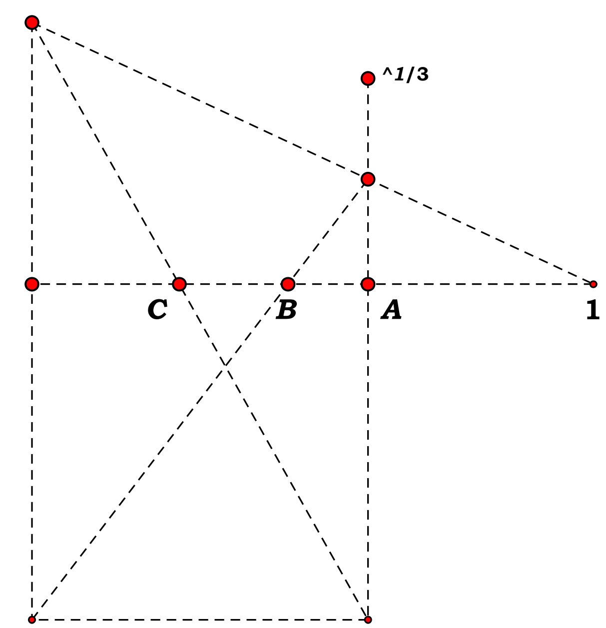
$$\frac{1}{1 - A} = 2.49299$$

$$\frac{1}{1 - B} = 1.83857$$

$$\frac{1}{1 - C} = 1.35594$$

$$\left(\frac{1}{1 - A}\right)^{\frac{2}{3}} - \frac{1}{1 - B} = 0.00000$$

$$\left(\frac{1}{1 - A}\right)^{\frac{1}{3}} - \frac{1}{1 - C} = 0.00000$$



Any Root 1-4

Thursday, November 18, 2021

Unit = 1.00000

A = 0.52124

B = 0.42444

C = 0.30807

D = 0.16818

$$\frac{1}{1-A} = 2.08872$$

$$\frac{1}{1-B} = 1.73744$$

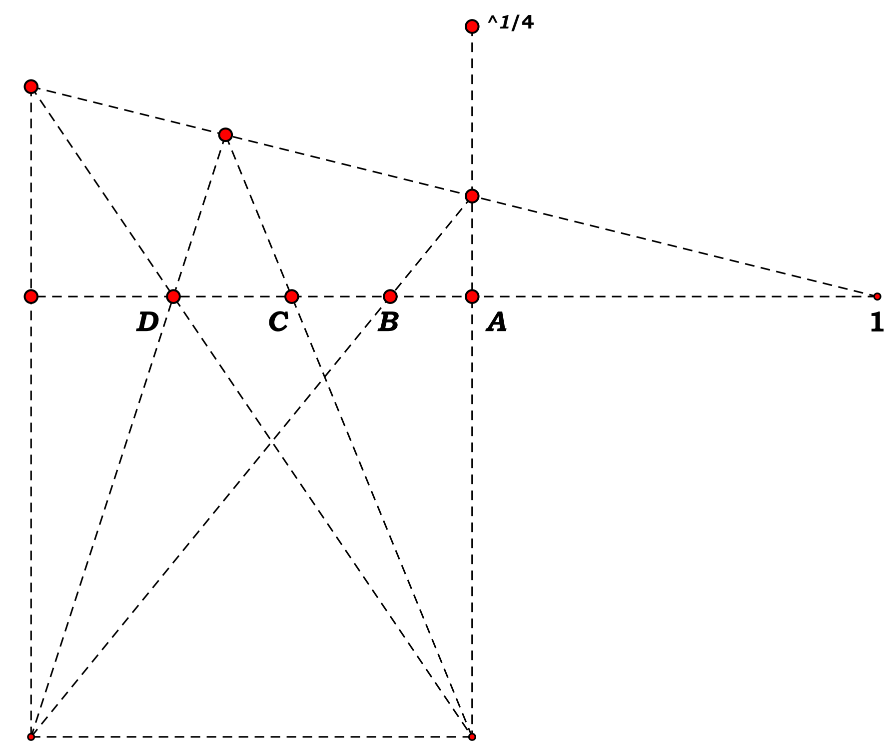
$$\frac{1}{1-C} = 1.44524$$

$$\frac{1}{1-D} = 1.20218$$

$$\left(\frac{1}{1-A}\right)^{\frac{3}{4}} - \frac{1}{1-B} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{2}{4}} - \frac{1}{1-C} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{1}{4}} - \frac{1}{1-D} = 0.00000$$



Any Root 1-5

Thursday, November 18, 2021

$Unit = 1.00000$

$A = 0.63095$

$B = 0.54952$

$C = 0.45014$

$D = 0.32882$

$E = 0.18075$

$$\frac{1}{1-A} = 2.70963$$

$$\frac{1}{1-B} = 2.21987$$

$$\frac{1}{1-C} = 1.81864$$

$$\frac{1}{1-D} = 1.48992$$

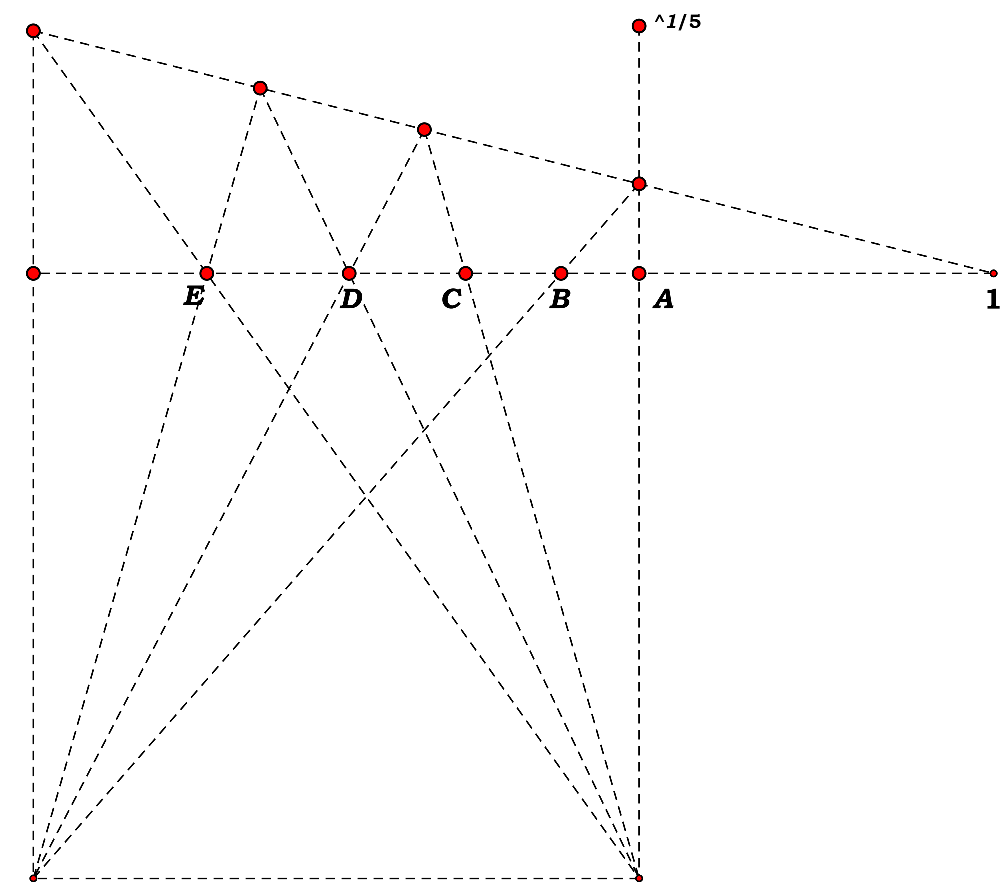
$$\frac{1}{1-E} = 1.22062$$

$$\left(\frac{1}{1-A}\right)^{\frac{4}{5}} - \frac{1}{1-B} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{3}{5}} - \frac{1}{1-C} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{2}{5}} - \frac{1}{1-D} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{1}{5}} - \frac{1}{1-E} = 0.00000$$



Any Root 1-6

Thursday, November 18, 2021

$\boldsymbol{Unit}$ = 1.00000

A = 0.82515

B = 0.76617

C = 0.68731

D = 0.58184

E = 0.44081

F = 0.25221

$$\frac{1}{1-A} = 5.71903$$

$$\frac{1}{1-B} = 4.27664$$

$$\frac{1}{1-C} = 3.19802$$

$$\frac{1}{1-D} = 2.39145$$

$$\frac{1}{1-E} = 1.78830$$

$$\frac{1}{1-F} = 1.33727$$

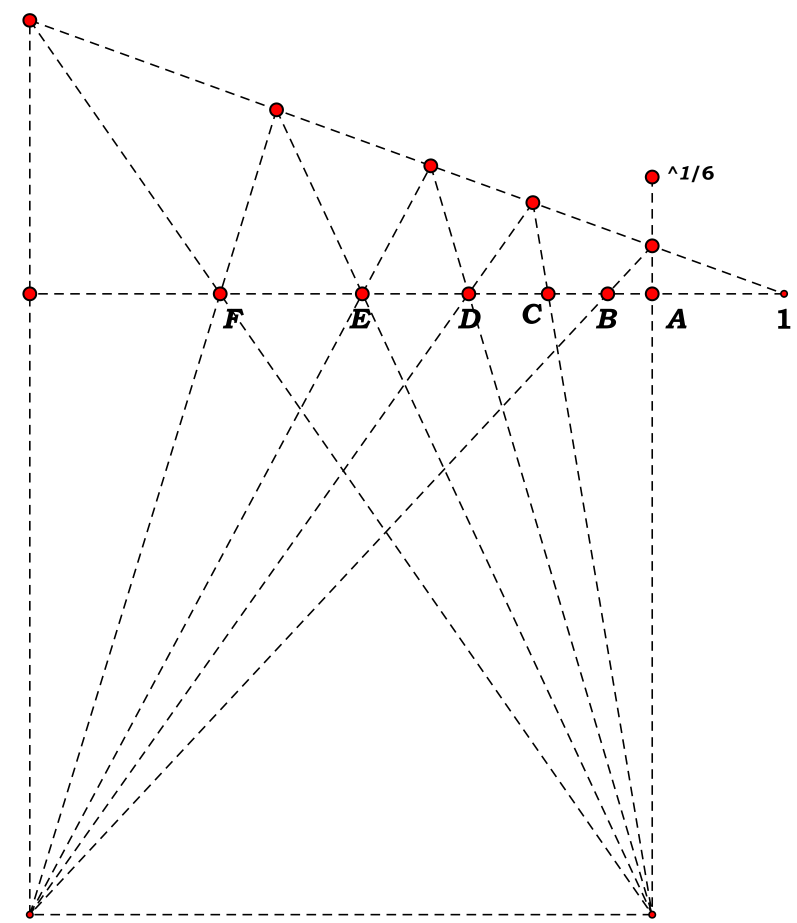
$$\left(\frac{1}{1-A}\right)^{\frac{5}{6}} - \frac{1}{1-B} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{4}{6}} - \frac{1}{1-C} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{3}{6}} - \frac{1}{1-D} = 0.00000$$

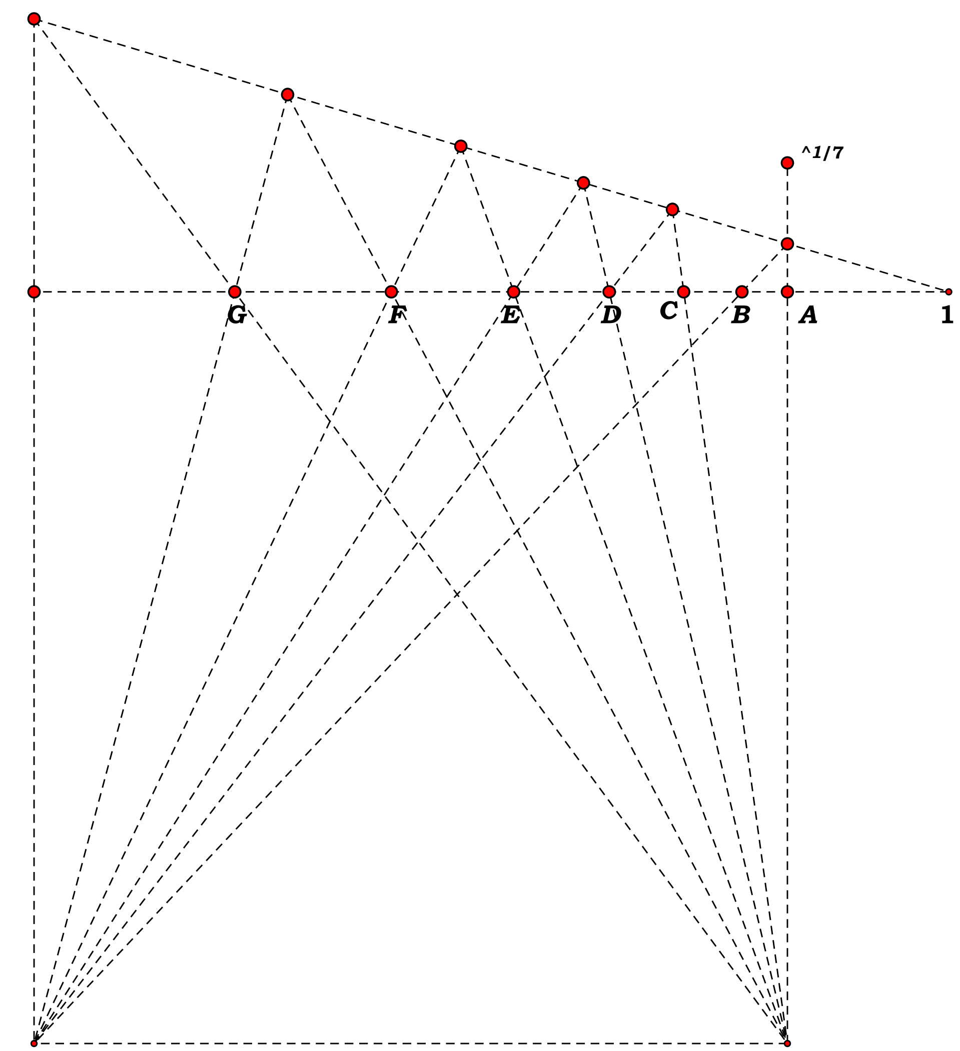
$$\left(\frac{1}{1-A}\right)^{\frac{2}{6}} - \frac{1}{1-E} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{1}{6}} - \frac{1}{1-F} = 0.00000$$



Any Root 1-7

Thursday, November 18, 2021



Unit **= 1.00000**

A **= 0.82344**

B **= 0.77381**

C **= 0.71022**

D **= 0.62876**

E **= 0.52440**

F **= 0.39071**

G **= 0.21943**

$$\frac{1}{1-A} = 5.66381 \qquad \left(\frac{1}{1-A}\right)^{\frac{6}{7}} - \frac{1}{1-B} = 0.00000$$

$$\frac{1}{1-B} = 4.42101 \qquad \left(\frac{1}{1-A}\right)^{\frac{5}{7}} - \frac{1}{1-C} = 0.00000$$

$$\frac{1}{1-C} = 3.45092 \qquad \left(\frac{1}{1-A}\right)^{\frac{4}{7}} - \frac{1}{1-D} = 0.00000$$

$$\frac{1}{1-D} = 2.69369 \qquad \left(\frac{1}{1-A}\right)^{\frac{3}{7}} - \frac{1}{1-E} = 0.00000$$

$$\frac{1}{1-E} = 2.10262 \qquad \left(\frac{1}{1-A}\right)^{\frac{2}{7}} - \frac{1}{1-F} = 0.00000$$

$$\frac{1}{1-F} = 1.64125 \qquad \left(\frac{1}{1-A}\right)^{\frac{1}{7}} - \frac{1}{1-G} = 0.00000$$

$$\frac{1}{1-G} = 1.28111$$

Any Root 1-8

Thursday, November 18, 2021

$Unit$ = 1.00000
A = 0.80013
B = 0.75557
C = 0.70108
D = 0.63443
E = 0.55293
F = 0.45326
G = 0.33137
H = 0.18230

$\frac{1}{1-A} = 5.00330$

$\frac{1}{1-B} = 4.09118$

$\frac{1}{1-C} = 3.34535$

$\frac{1}{1-D} = 2.73549$

$\frac{1}{1-E} = 2.23680$

$\frac{1}{1-F} = 1.82903$

$\frac{1}{1-G} = 1.49560$

$\frac{1}{1-H} = 1.22295$

$$\left(\frac{1}{1-A}\right)^{\frac{7}{8}} - \frac{1}{1-B} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{6}{8}} - \frac{1}{1-C} = 0.00000$$

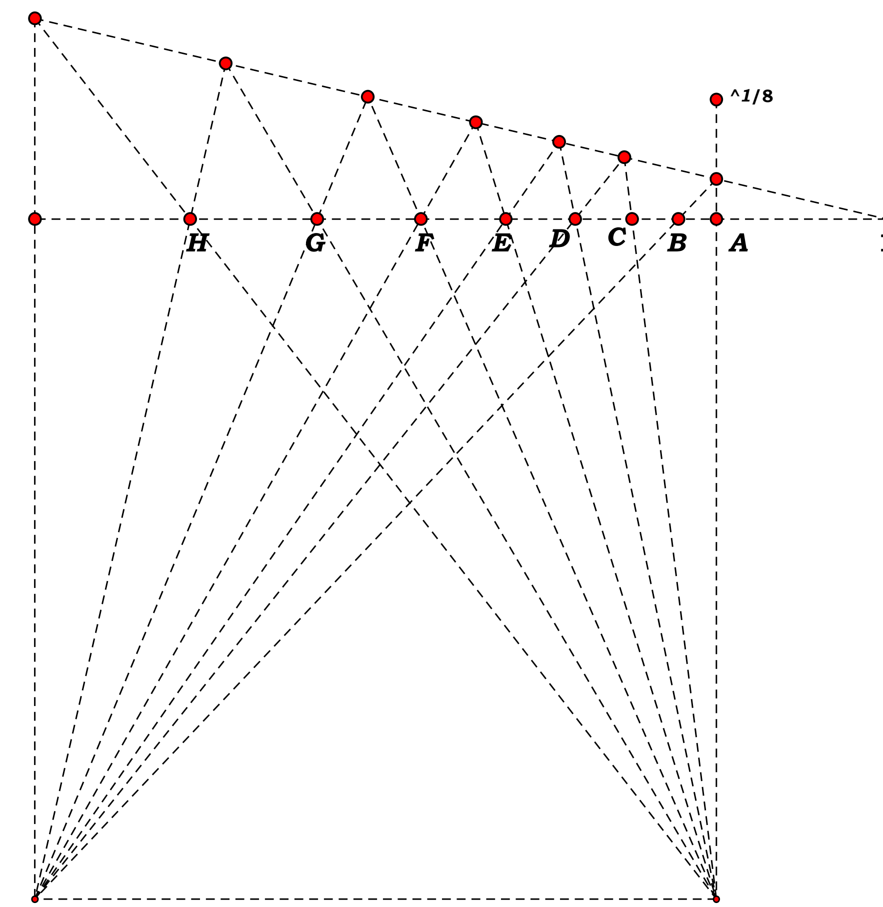
$$\left(\frac{1}{1-A}\right)^{\frac{5}{8}} - \frac{1}{1-D} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{4}{8}} - \frac{1}{1-E} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{3}{8}} - \frac{1}{1-F} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{2}{8}} - \frac{1}{1-G} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{1}{8}} - \frac{1}{1-H} = 0.00000$$



Any Root 1-9

Thursday, November 18, 2021

$Unit$ = 1.00000
A = 0.74151
B = 0.69958
C = 0.65085
D = 0.59421
E = 0.52839
F = 0.45189
G = 0.36298
H = 0.25966
I = 0.13957

$$\frac{1}{1-A} = 3.86857 \qquad \left(\frac{1}{1-A}\right)^{\frac{8}{9}} - \frac{1}{1-B} = 0.00000$$

$$\frac{1}{1-B} = 3.32864 \qquad \left(\frac{1}{1-A}\right)^{\frac{7}{9}} - \frac{1}{1-C} = 0.00000$$

$$\frac{1}{1-C} = 2.86407 \qquad \left(\frac{1}{1-A}\right)^{\frac{6}{9}} - \frac{1}{1-D} = 0.00000$$

$$\frac{1}{1-D} = 2.46434 \qquad \left(\frac{1}{1-A}\right)^{\frac{5}{9}} - \frac{1}{1-E} = 0.00000$$

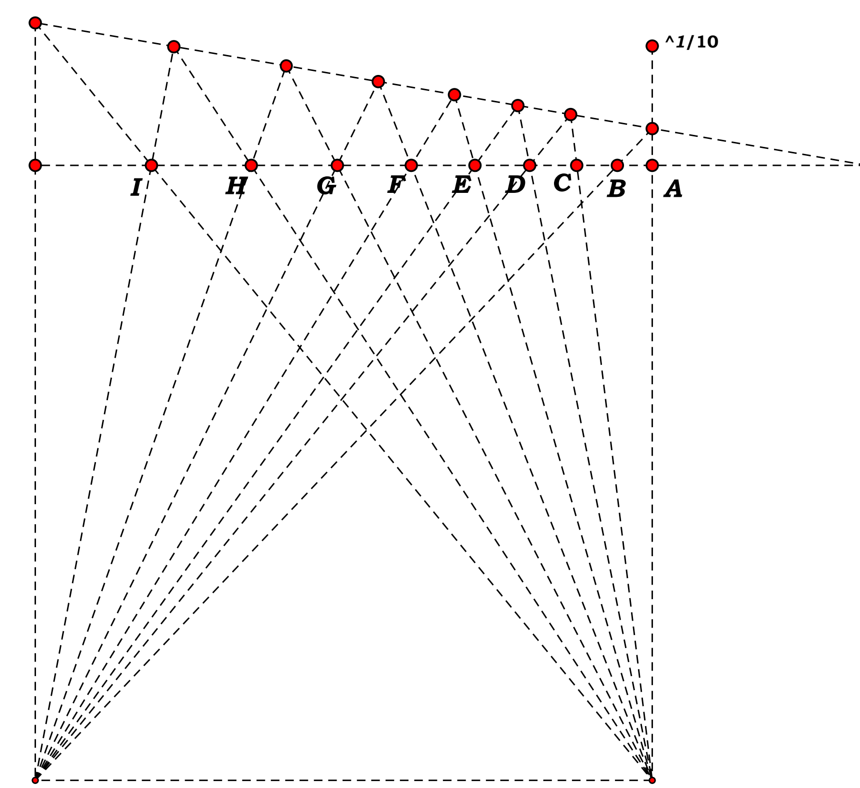
$$\frac{1}{1-E} = 2.12040 \qquad \left(\frac{1}{1-A}\right)^{\frac{4}{9}} - \frac{1}{1-F} = 0.00000$$

$$\frac{1}{1-F} = 1.82446 \qquad \left(\frac{1}{1-A}\right)^{\frac{3}{9}} - \frac{1}{1-G} = 0.00000$$

$$\frac{1}{1-G} = 1.56982 \qquad \left(\frac{1}{1-A}\right)^{\frac{2}{9}} - \frac{1}{1-H} = 0.00000$$

$$\frac{1}{1-H} = 1.35072 \qquad \left(\frac{1}{1-A}\right)^{\frac{1}{9}} - \frac{1}{1-I} = 0.00000$$

$$\frac{1}{1-I} = 1.16221$$



Any Root 1-10

Thursday, November 18, 2021

$\boldsymbol{Unit} = 1.00000$

$A = 0.83709$

$\frac{1}{1-A} = 6.13838$

$B = 0.80468$

$\frac{1}{1-B} = 5.11974$

$\left(\frac{1}{1-A}\right)^{\frac{9}{10}} - \frac{1}{1-B} = 0.00000$

$C = 0.76582$

$D = 0.71922$

$E = 0.66336$

$F = 0.59638$

$G = 0.51607$

$H = 0.41979$

$I = 0.30435$

$J = 0.16595$

$\frac{1}{1-C} = 4.27015$

$\frac{1}{1-D} = 3.56154$

$\frac{1}{1-E} = 2.97052$

$\frac{1}{1-F} = 2.47758$

$\frac{1}{1-G} = 2.06643$

$\frac{1}{1-H} = 1.72352$

$\frac{1}{1-I} = 1.43751$

$\frac{1}{1-J} = 1.19896$

$\left(\frac{1}{1-A}\right)^{\frac{8}{10}} - \frac{1}{1-C} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{7}{10}} - \frac{1}{1-D} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{6}{10}} - \frac{1}{1-E} = 0.00000$

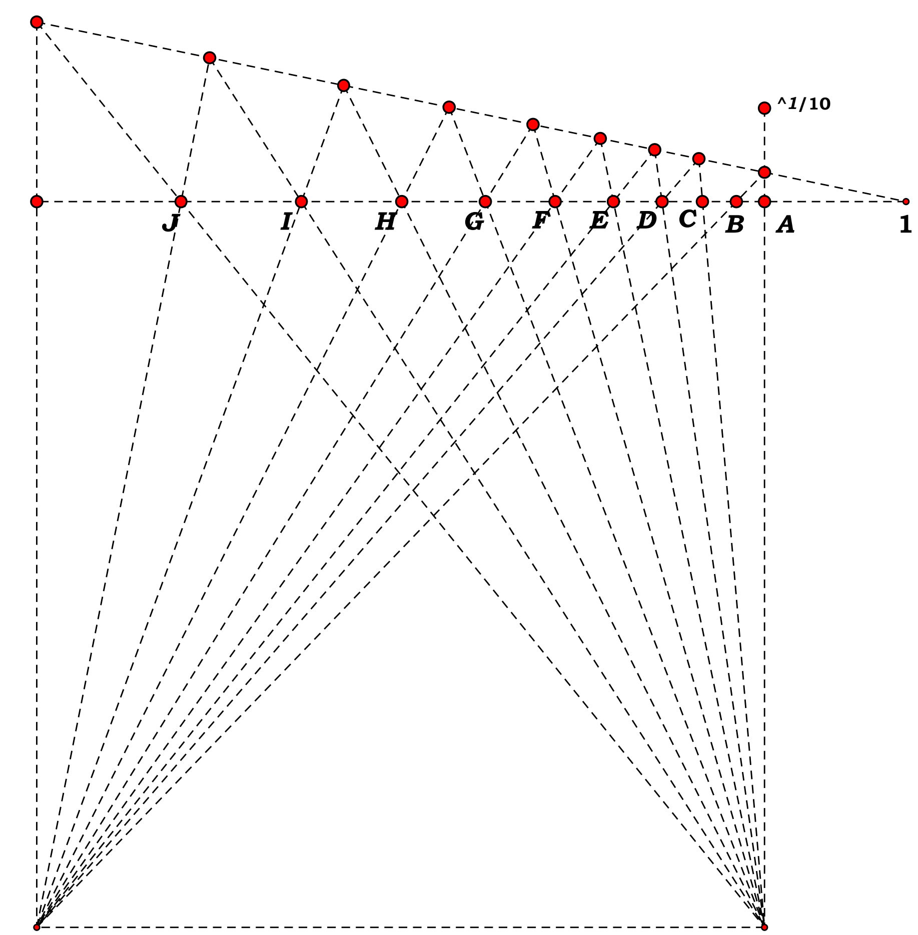
$\left(\frac{1}{1-A}\right)^{\frac{5}{10}} - \frac{1}{1-F} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{4}{10}} - \frac{1}{1-G} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{3}{10}} - \frac{1}{1-H} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{2}{10}} - \frac{1}{1-I} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{1}{10}} - \frac{1}{1-J} = 0.00000$



Any Root 1-11

Thursday, November 18, 2021

$A = 0.85344$

$\frac{1}{1-A} = 6.82294$

$B = 0.82548$

$\frac{1}{1-B} = 5.73002$

$\left(\frac{1}{1-A}\right)^{\frac{10}{11}} - \frac{1}{1-B} = 0.00000$

$C = 0.79219$

$D = 0.75256$

$E = 0.70536$

$F = 0.64916$

$G = 0.58224$

$H = 0.50256$

$I = 0.40768$

$J = 0.29471$

$K = 0.16018$

$\frac{1}{1-C} = 4.81217$

$\frac{1}{1-D} = 4.04134$

$\frac{1}{1-E} = 3.39398$

$\frac{1}{1-F} = 2.85032$

$\frac{1}{1-G} = 2.39375$

$\frac{1}{1-H} = 2.01031$

$\frac{1}{1-I} = 1.68829$

$\frac{1}{1-J} = 1.41785$

$\frac{1}{1-K} = 1.19074$

$\left(\frac{1}{1-A}\right)^{\frac{9}{11}} - \frac{1}{1-C} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{8}{11}} - \frac{1}{1-D} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{7}{11}} - \frac{1}{1-E} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{6}{11}} - \frac{1}{1-F} = 0.00000$

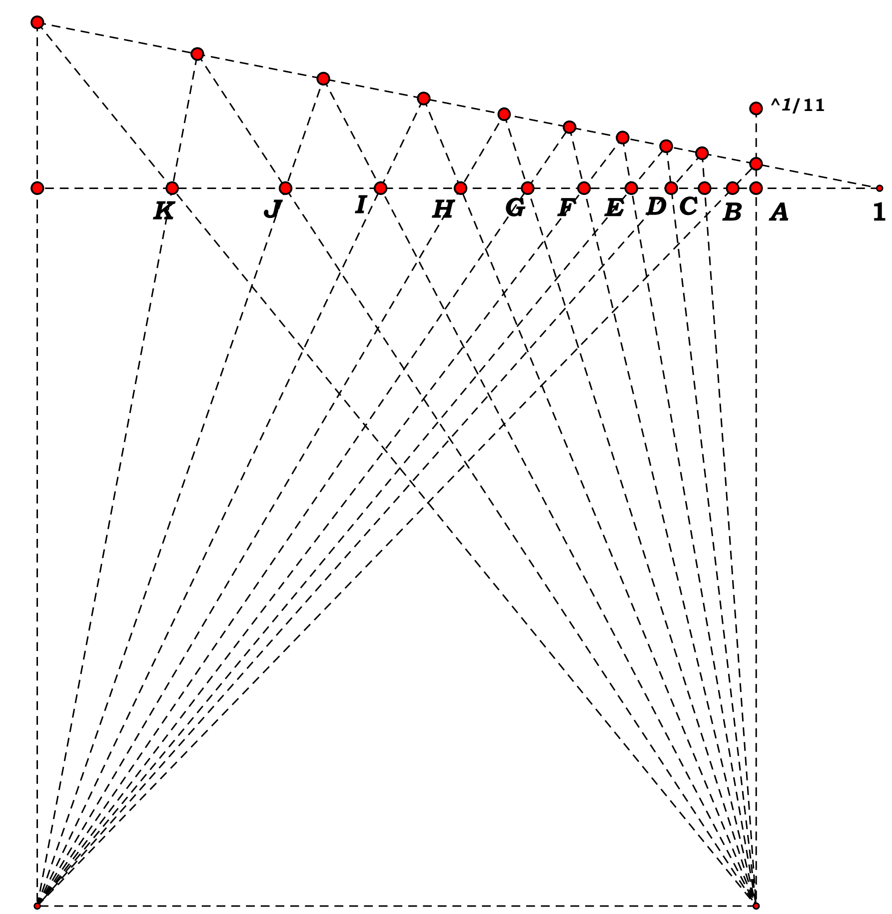
$\left(\frac{1}{1-A}\right)^{\frac{5}{11}} - \frac{1}{1-G} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{4}{11}} - \frac{1}{1-H} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{3}{11}} - \frac{1}{1-I} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{2}{11}} - \frac{1}{1-J} = 0.00000$

$\left(\frac{1}{1-A}\right)^{\frac{1}{11}} - \frac{1}{1-K} = 0.00000$



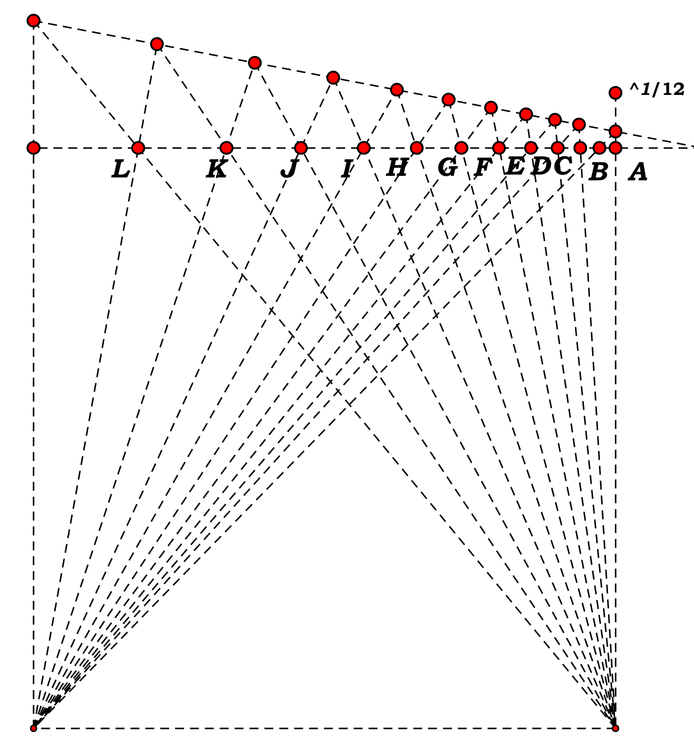
Any Root 1-12

Thursday, November 18, 2021

Unit = 1.00000
A = 0.87032
B = 0.84626
C = 0.81773
D = 0.78390
E = 0.74380
F = 0.69626
G = 0.63989
H = 0.57307
I = 0.49384
J = 0.39991
K = 0.28855
L = 0.15653

$$\frac{1}{1-A} = 7.71141$$
$$\frac{1}{1-B} = 6.50438$$
$$\frac{1}{1-C} = 5.48628$$
$$\frac{1}{1-D} = 4.62754$$
$$\frac{1}{1-E} = 3.90321$$
$$\frac{1}{1-F} = 3.29226$$
$$\frac{1}{1-G} = 2.77694$$
$$\frac{1}{1-H} = 2.34228$$
$$\frac{1}{1-I} = 1.97566$$
$$\frac{1}{1-J} = 1.66642$$
$$\frac{1}{1-K} = 1.40558$$
$$\frac{1}{1-L} = 1.18557$$

$$\left(\frac{1}{1-A}\right)^{\frac{11}{12}} - \frac{1}{1-B} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{10}{12}} - \frac{1}{1-C} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{9}{12}} - \frac{1}{1-D} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{8}{12}} - \frac{1}{1-E} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{7}{12}} - \frac{1}{1-F} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{6}{12}} - \frac{1}{1-G} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{5}{12}} - \frac{1}{1-H} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{4}{12}} - \frac{1}{1-I} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{3}{12}} - \frac{1}{1-J} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{2}{12}} - \frac{1}{1-K} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{1}{12}} - \frac{1}{1-L} = 0.00000$$



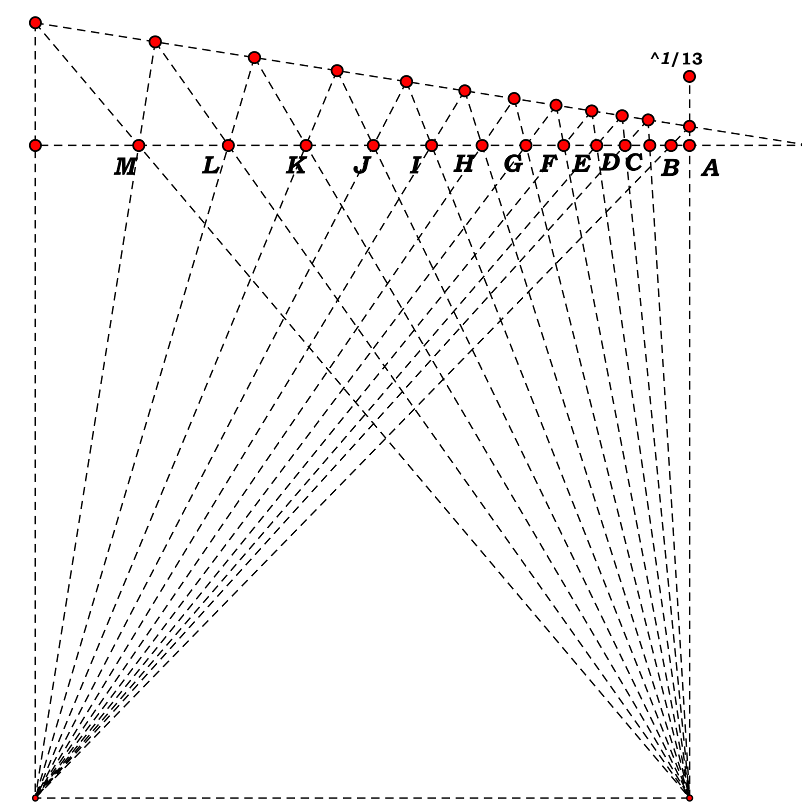
Any Root 1-13

Thursday, November 18, 2021

$A = 0.84512$
$B = 0.82123$
$C = 0.79365$
$D = 0.76181$
$E = 0.72507$
$F = 0.68265$
$G = 0.63369$
$H = 0.57718$
$I = 0.51196$
$J = 0.43666$
$K = 0.34976$
$L = 0.24944$
$M = 0.13365$

$\frac{1}{1-A} = 6.45661$
$\frac{1}{1-B} = 5.59366$
$\frac{1}{1-C} = 4.84605$
$\frac{1}{1-D} = 4.19837$
$\frac{1}{1-E} = 3.63724$
$\frac{1}{1-F} = 3.15111$
$\frac{1}{1-G} = 2.72996$
$\frac{1}{1-H} = 2.36509$
$\frac{1}{1-I} = 2.04899$
$\frac{1}{1-J} = 1.77514$
$\frac{1}{1-K} = 1.53789$
$\frac{1}{1-L} = 1.33234$
$\frac{1}{1-M} = 1.15427$

$$\left(\frac{1}{1-A}\right)^{\frac{12}{13}} - \frac{1}{1-B} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{11}{13}} - \frac{1}{1-C} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{10}{13}} - \frac{1}{1-D} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{9}{13}} - \frac{1}{1-E} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{8}{13}} - \frac{1}{1-F} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{7}{13}} - \frac{1}{1-G} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{6}{13}} - \frac{1}{1-H} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{5}{13}} - \frac{1}{1-I} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{4}{13}} - \frac{1}{1-J} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{3}{13}} - \frac{1}{1-K} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{2}{13}} - \frac{1}{1-L} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{1}{13}} - \frac{1}{1-M} = 0.00000$$



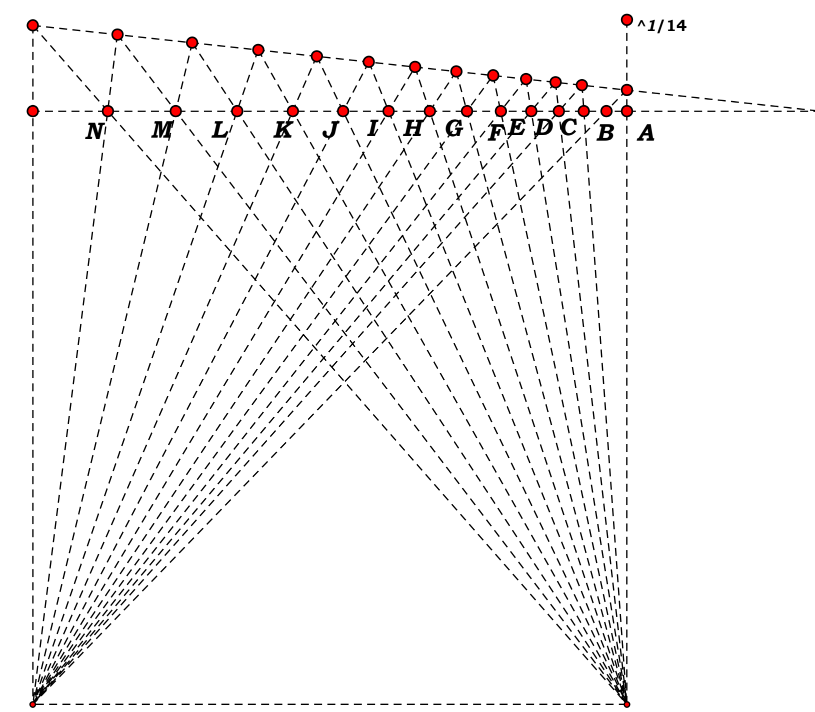
Any Root 1-14

Thursday, November 18, 2021

$Unit$ = 1.00000
A = 0.75337
B = 0.72743
C = 0.69877
D = 0.66709
E = 0.63209
F = 0.59340
G = 0.55064
H = 0.50338
I = 0.45116
J = 0.39344
K = 0.32965
L = 0.25916
M = 0.18125
N = 0.09515

$$\frac{1}{1-A} = 4.05466$$
$$\frac{1}{1-B} = 3.66884$$
$$\frac{1}{1-C} = 3.31974$$
$$\frac{1}{1-D} = 3.00385$$
$$\frac{1}{1-E} = 2.71802$$
$$\frac{1}{1-F} = 2.45939$$
$$\frac{1}{1-G} = 2.22537$$
$$\frac{1}{1-H} = 2.01362$$
$$\frac{1}{1-I} = 1.82202$$
$$\frac{1}{1-J} = 1.64864$$
$$\frac{1}{1-K} = 1.49177$$
$$\frac{1}{1-L} = 1.34982$$
$$\frac{1}{1-M} = 1.22138$$
$$\frac{1}{1-N} = 1.10516$$

$$\left(\frac{1}{1-A}\right)^{\frac{13}{14}} - \frac{1}{1-B} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{12}{14}} - \frac{1}{1-C} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{11}{14}} - \frac{1}{1-D} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{10}{14}} - \frac{1}{1-E} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{9}{14}} - \frac{1}{1-F} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{8}{14}} - \frac{1}{1-G} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{7}{14}} - \frac{1}{1-H} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{6}{14}} - \frac{1}{1-I} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{5}{14}} - \frac{1}{1-J} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{4}{14}} - \frac{1}{1-K} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{3}{14}} - \frac{1}{1-L} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{2}{14}} - \frac{1}{1-M} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{1}{14}} - \frac{1}{1-N} = 0.00000$$



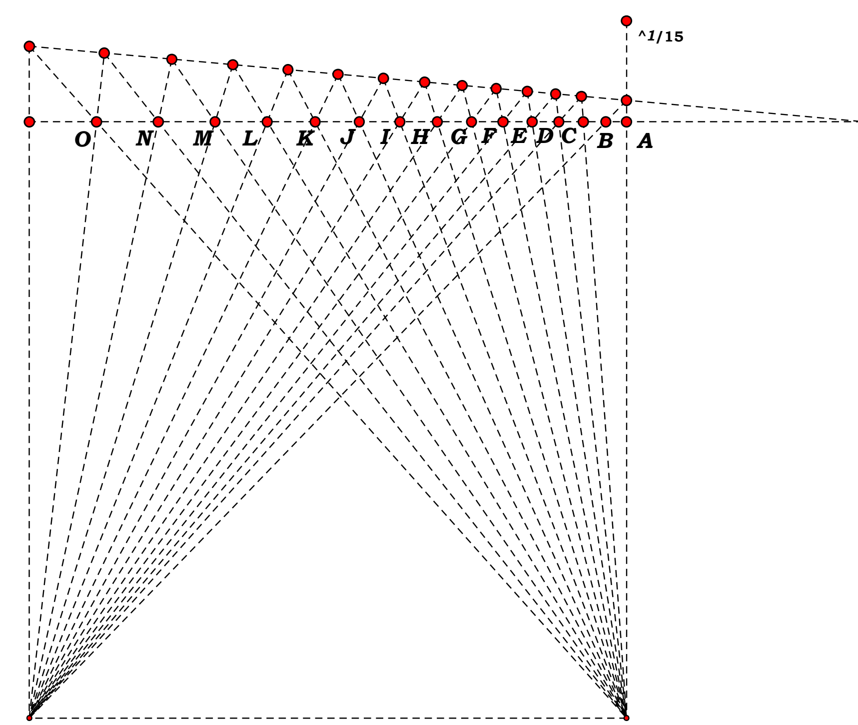
Any Root 1-15

Thursday, November 18, 2021

$$\left(\frac{1}{1-A}\right)^{\frac{14}{15}} - \frac{1}{1-B} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{13}{15}} - \frac{1}{1-C} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{12}{15}} - \frac{1}{1-D} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{11}{15}} - \frac{1}{1-E} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{10}{15}} - \frac{1}{1-F} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{9}{15}} - \frac{1}{1-G} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{8}{15}} - \frac{1}{1-H} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{7}{15}} - \frac{1}{1-I} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{6}{15}} - \frac{1}{1-J} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{5}{15}} - \frac{1}{1-K} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{4}{15}} - \frac{1}{1-L} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{3}{15}} - \frac{1}{1-M} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{2}{15}} - \frac{1}{1-N} = 0.00000$$
$$\left(\frac{1}{1-A}\right)^{\frac{1}{15}} - \frac{1}{1-O} = 0.00000$$

$Unit$ = 1.00000
A = 0.71622
B = 0.69136
C = 0.66433
D = 0.63493
E = 0.60295
F = 0.56816
G = 0.53034
H = 0.48920
I = 0.44445
J = 0.39579
K = 0.34286
L = 0.28529
M = 0.22269
N = 0.15460
O = 0.08054

$$\frac{1}{1-A} = 3.52389$$
$$\frac{1}{1-B} = 3.24007$$
$$\frac{1}{1-C} = 2.97911$$
$$\frac{1}{1-D} = 2.73916$$
$$\frac{1}{1-E} = 2.51855$$
$$\frac{1}{1-F} = 2.31570$$
$$\frac{1}{1-G} = 2.12919$$
$$\frac{1}{1-H} = 1.95770$$
$$\frac{1}{1-I} = 1.80002$$
$$\frac{1}{1-J} = 1.65504$$
$$\frac{1}{1-K} = 1.52174$$
$$\frac{1}{1-L} = 1.39918$$
$$\frac{1}{1-M} = 1.28648$$
$$\frac{1}{1-N} = 1.18287$$
$$\frac{1}{1-O} = 1.08760$$



Any Power P1.

Wednesday, November 17, 2021

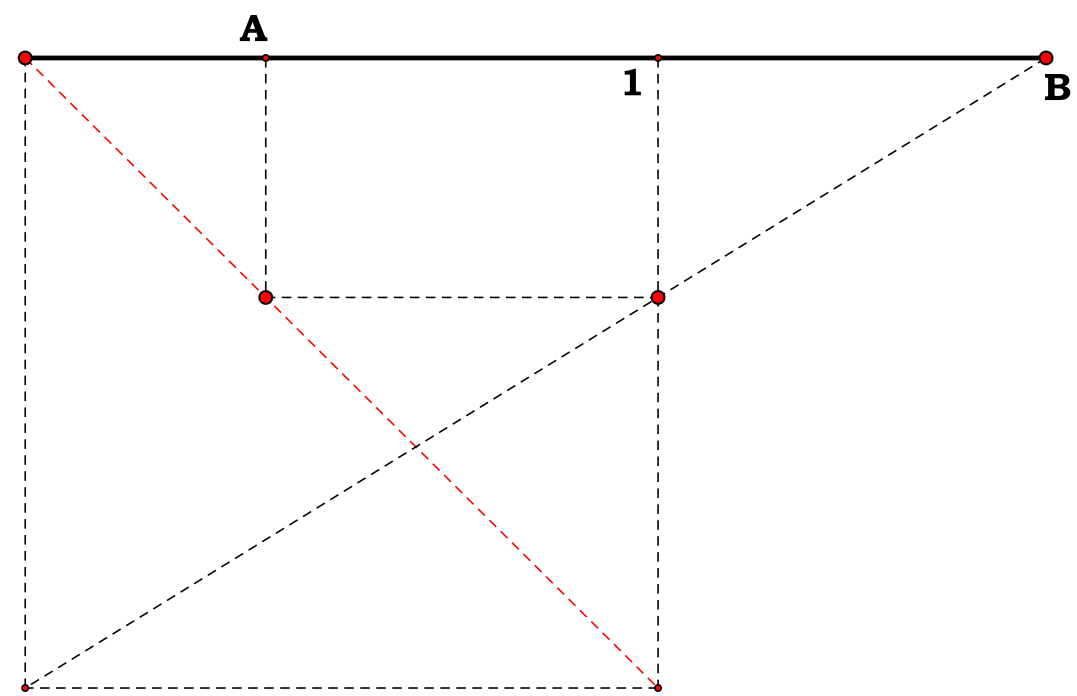
The base figure for these demonstrations shows the initial values for induction at B and deduction at A, i.e. Arithmetic progression from B and Geometric progression from A. Arithmetic progression is afforded by the recursion of the unit infinitely from 1, while Geometric Progression recursively provides the ratio between 0 and 1. Thus, arithmetically we cannot visualize the infinite set of all numbers, but geometrically we can.

Unit = 1.00000

A = 0.38009

$$\frac{1}{1 - A} = 1.61313$$

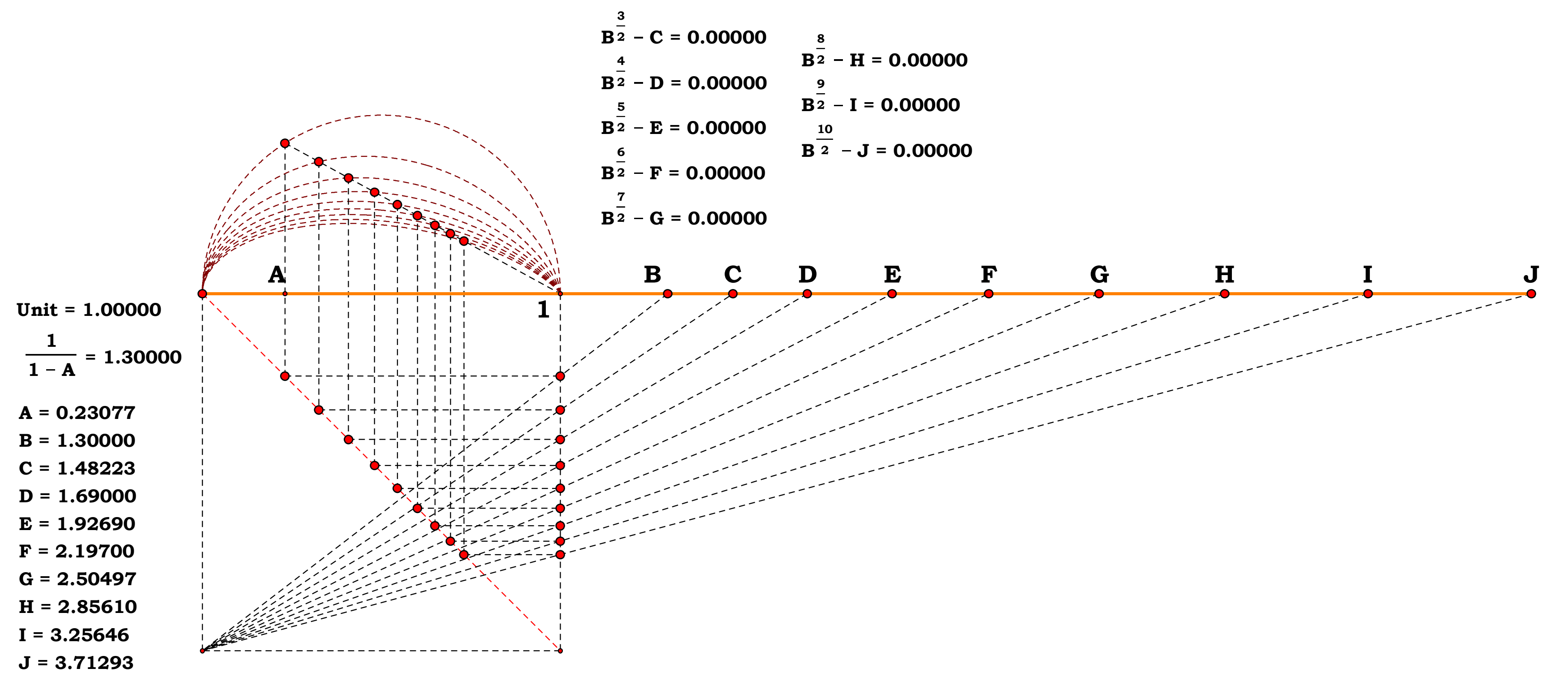
B = 1.61313



Any Power P2

Wednesday, November 17, 2021

Here is a what I have called a 2 prime exponential series.



Any Power P3

Wednesday, November 17, 2021

A third root series.

$B^{\frac{2}{3}} - C = 0.00000$

$B^{\frac{4}{3}} - D = 0.00000$

$B^{\frac{5}{3}} - E = 0.00000$

$B^{\frac{6}{3}} - F = 0.00000$

$B^{\frac{7}{3}} - G = 0.00000$

$B^{\frac{8}{3}} - H = 0.00000$

$B^{\frac{9}{3}} - I = 0.00000$

$B^{\frac{10}{3}} - J = 0.00000$

$A = 0.23077$

$\frac{1}{1 - A} = 1.30000$

$B = 1.30000$

$C = 1.19114$

$D = 1.41881$

$E = 1.54848$

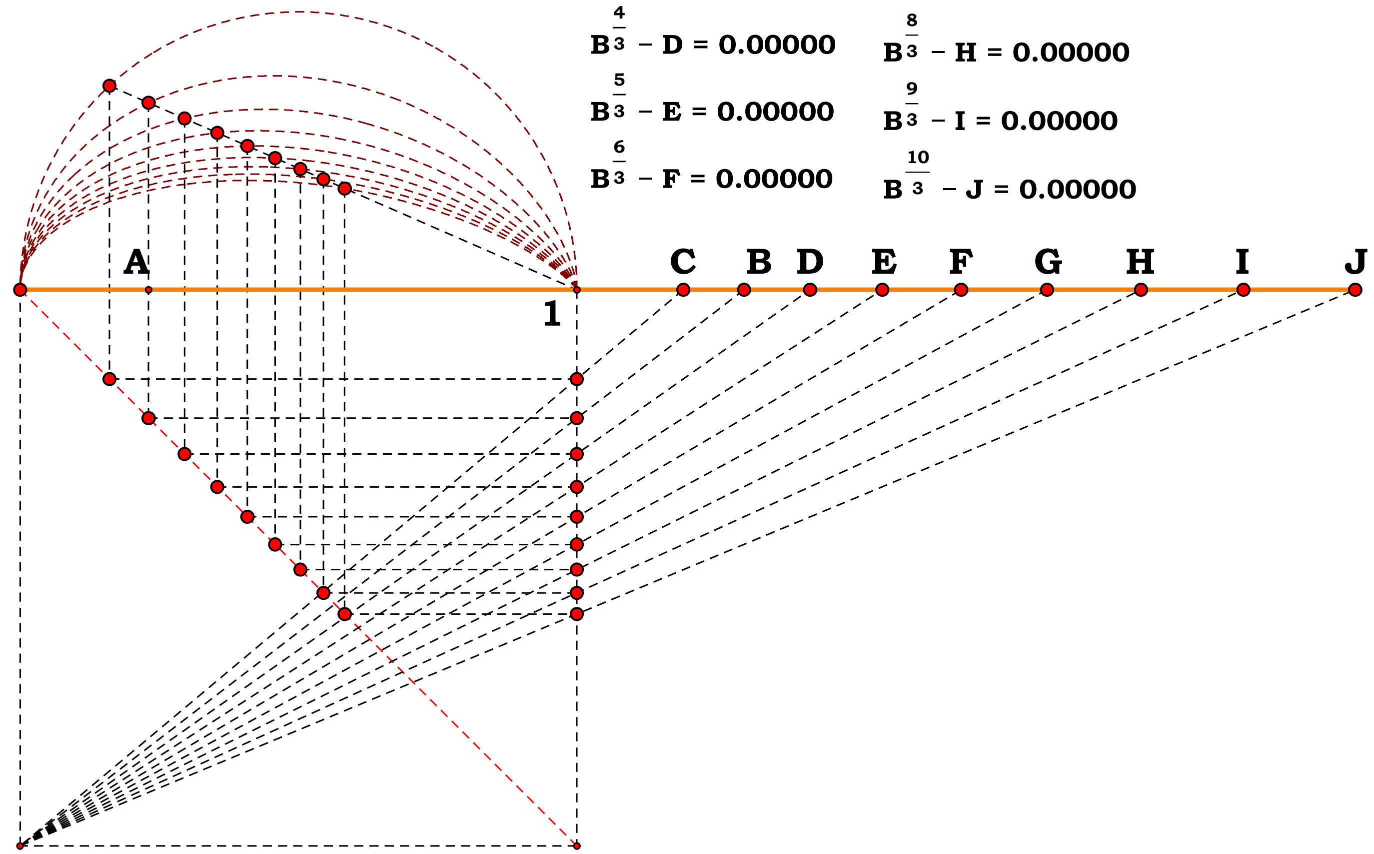
$F = 1.69000$

$G = 1.84445$

$H = 2.01302$

$I = 2.19700$

$J = 2.39779$



Any Power P4

Wednesday, November 17, 2021

Quarter Root series.

$A = 0.23077$

$\frac{1}{1-A} = 1.30000$

$B = 1.30000$

$C = 1.14018$

$D = 1.21747$

$E = 1.38813$

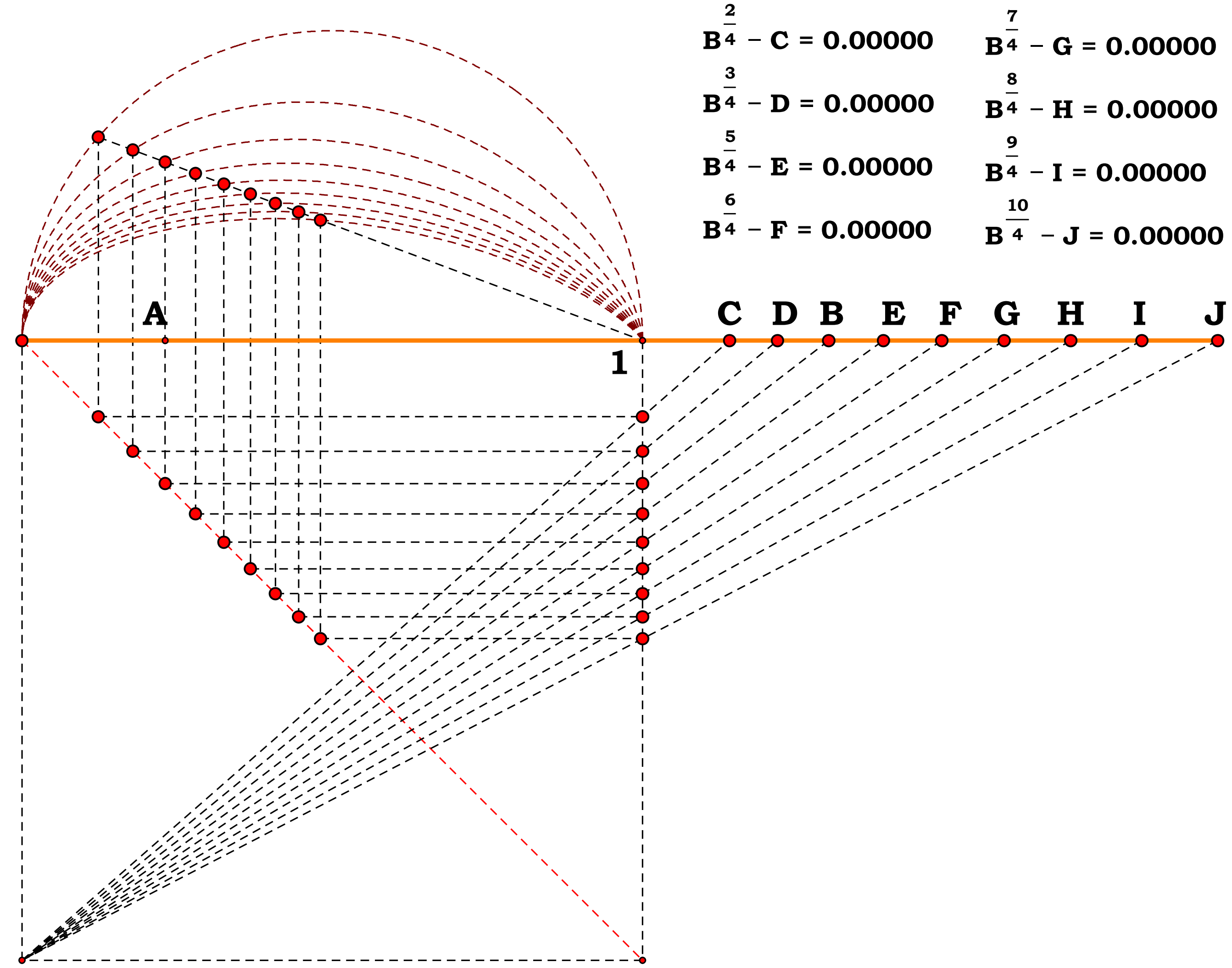
$F = 1.48223$

$G = 1.58271$

$H = 1.69000$

$I = 1.80457$

$J = 1.92690$



$B^{\frac{2}{4}} - C = 0.00000$

$B^{\frac{3}{4}} - D = 0.00000$

$B^{\frac{5}{4}} - E = 0.00000$

$B^{\frac{6}{4}} - F = 0.00000$

$B^{\frac{7}{4}} - G = 0.00000$

$B^{\frac{8}{4}} - H = 0.00000$

$B^{\frac{9}{4}} - I = 0.00000$

$B^{\frac{10}{4}} - J = 0.00000$

Any Power P5

Wednesday, November 17, 2021

$A = 0.23077$

$\dfrac{1}{1 - A} = 1.30000$

$B = 1.30000$

$C = 1.11065$

$D = 1.17049$

$E = 1.23354$

$F = 1.37004$

$G = 1.44385$

$H = 1.52163$

$I = 1.60361$

$J = 1.69000$

$B^{\frac{2}{5}} - C = 0.00000$

$B^{\frac{3}{5}} - D = 0.00000$

$B^{\frac{4}{5}} - E = 0.00000$

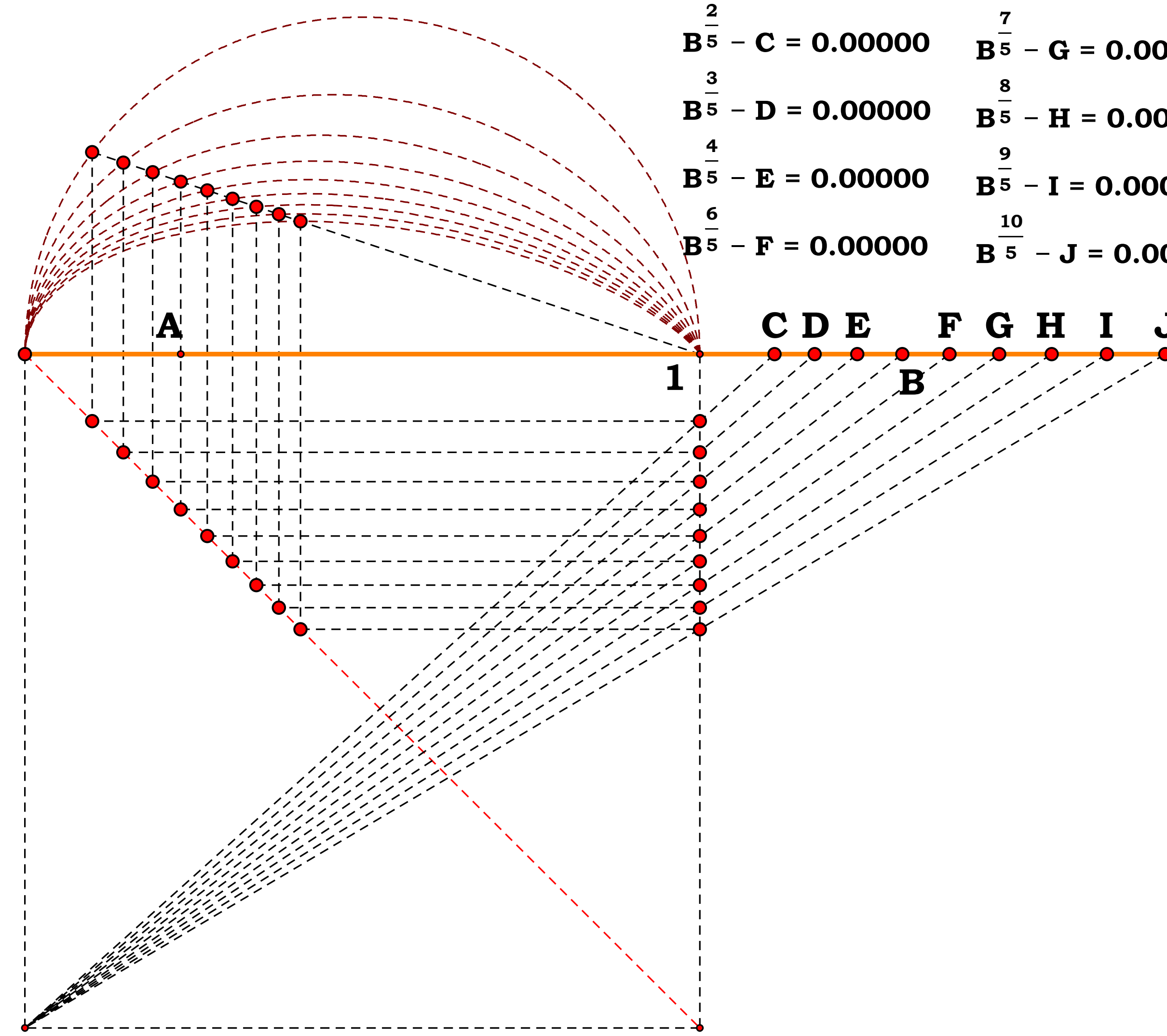
$B^{\frac{6}{5}} - F = 0.00000$

$B^{\frac{7}{5}} - G = 0.00000$

$B^{\frac{8}{5}} - H = 0.00000$

$B^{\frac{9}{5}} - I = 0.00000$

$B^{\frac{10}{5}} - J = 0.00000$



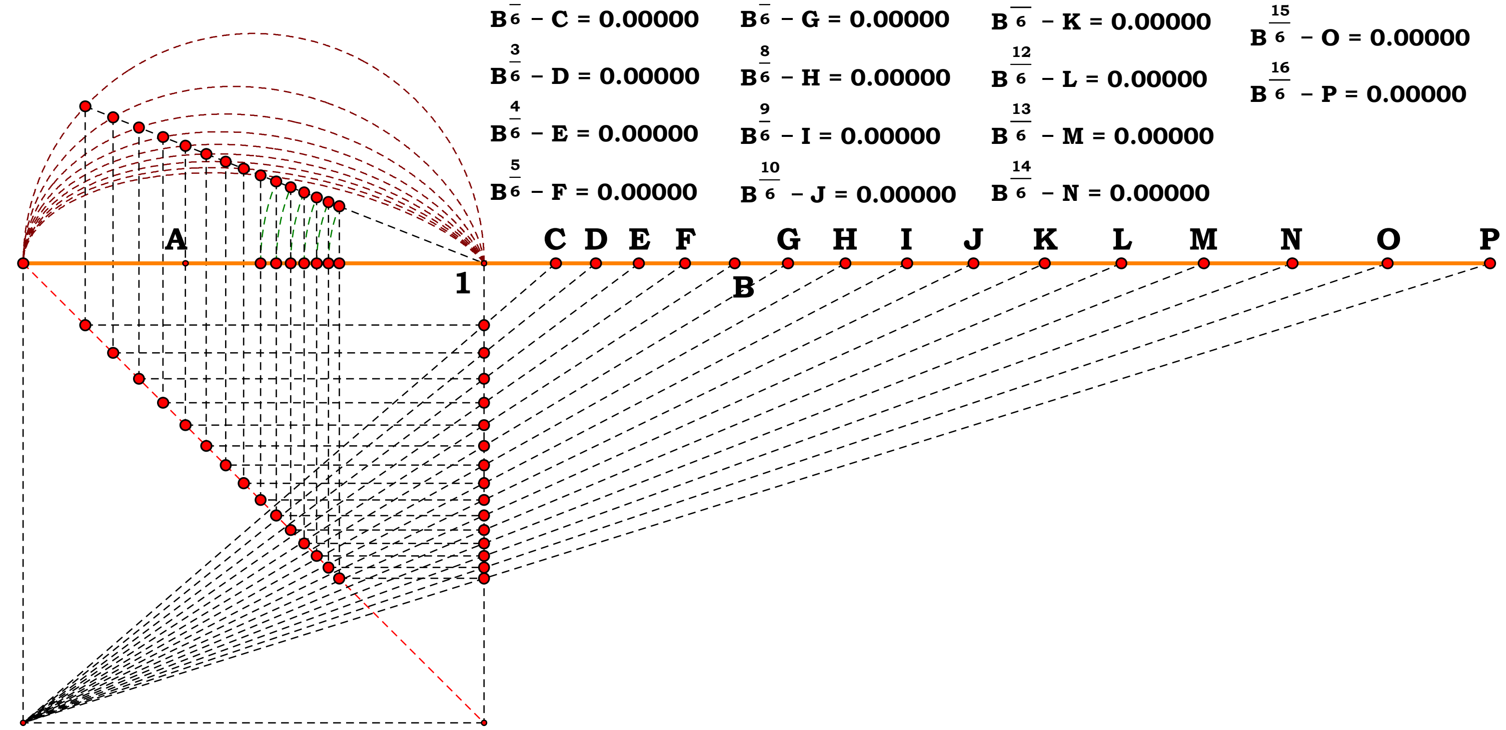
Any Power P6

Wednesday, November 17, 2021

A = 0.35215

$\frac{1}{1 - A} = 1.54358$

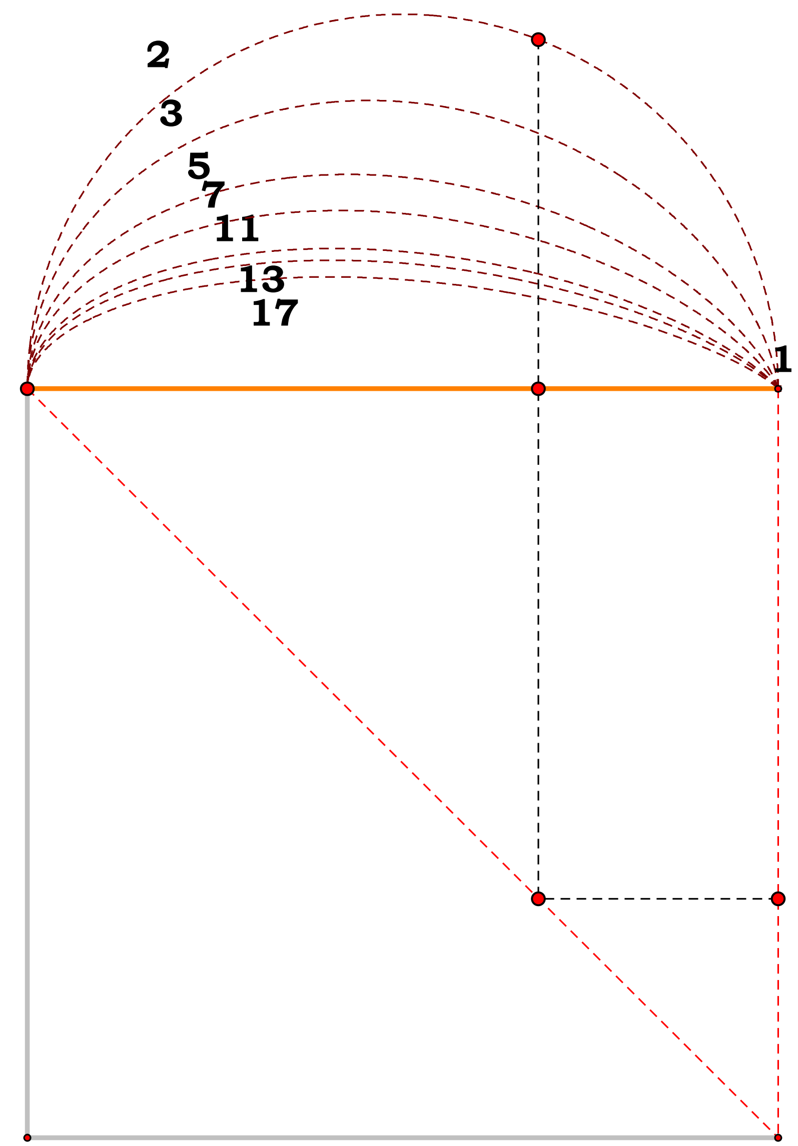
B = 1.54358
C = 1.15569
D = 1.24241
E = 1.33563
F = 1.43584
G = 1.65940
H = 1.78390
I = 1.91775
J = 2.06165
K = 2.21634
L = 2.38263
M = 2.56141
N = 2.75359
O = 2.96020
P = 3.18231



Multiplication

Thursday, November 18, 2021

One can use root loci which shows each one, but we learn in factoring to factor in accordance with prime numbers, therefore, one should construct a template of just prime numbers. How to use them, however, is a bit more complicated.



Multiplication 3x4

Thursday, November 18, 2021

We learn to effect our progression along a line by using two adjacent roots. When one uses two different roots, the product is a one multiplied by the other. Here is the third and fourth root progression from starting points. The result is the twelfth root series.

$N_1 = 8.00000$

A = 0.87500
B = 0.85135
C = 0.82322
D = 0.78978
E = 0.75000
F = 0.70270
G = 0.64645
H = 0.57955
I = 0.50000
J = 0.40540
K = 0.29289
L = 0.15910

$\frac{1}{1-A} = 8.00000$

$\frac{1}{1-B} = 6.72717$

$\frac{1}{1-C} = 5.65685$

$\frac{1}{1-D} = 4.75683$

$\frac{1}{1-E} = 4.00000$

$\frac{1}{1-F} = 3.36359$

$\frac{1}{1-G} = 2.82843$

$\frac{1}{1-H} = 2.37841$

$\frac{1}{1-I} = 2.00000$

$\frac{1}{1-J} = 1.68179$

$\frac{1}{1-K} = 1.41421$

$\frac{1}{1-L} = 1.18921$

$$\left(\frac{1}{1-A}\right)^{\frac{11}{12}} - \frac{1}{1-B} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{10}{12}} - \frac{1}{1-C} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{9}{12}} - \frac{1}{1-D} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{8}{12}} - \frac{1}{1-E} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{7}{12}} - \frac{1}{1-F} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{6}{12}} - \frac{1}{1-G} = 0.00000$$

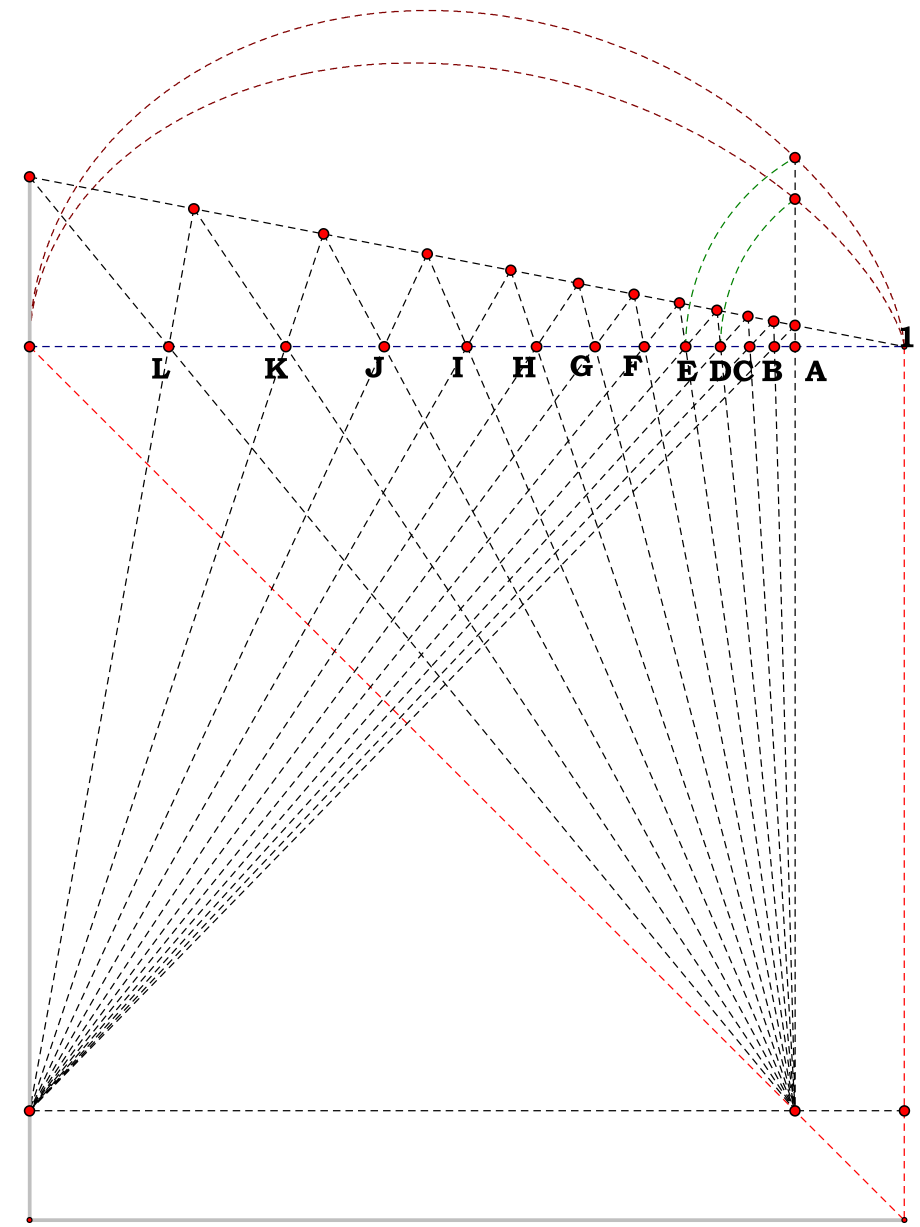
$$\left(\frac{1}{1-A}\right)^{\frac{5}{12}} - \frac{1}{1-H} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{4}{12}} - \frac{1}{1-I} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{3}{12}} - \frac{1}{1-J} = 0.00000$$

$$\left(\frac{1}{1-A}\right)^{\frac{2}{12}} - \frac{1}{1-K} = 0.00000$$

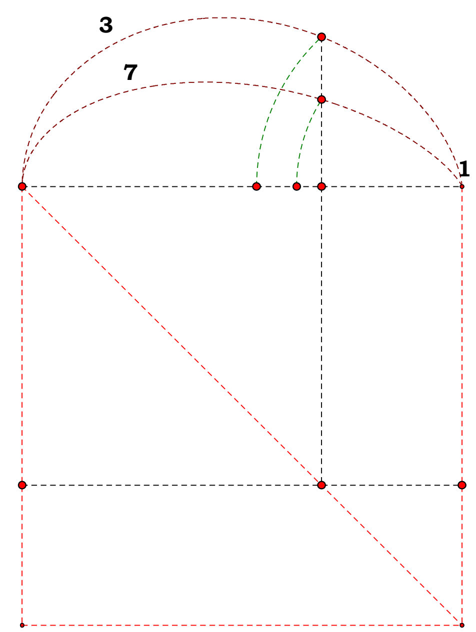
$$\left(\frac{1}{1-A}\right)^{\frac{1}{12}} - \frac{1}{1-L} = 0.00000$$



Multiplication 3x7 1

Thursday, November 18, 2021

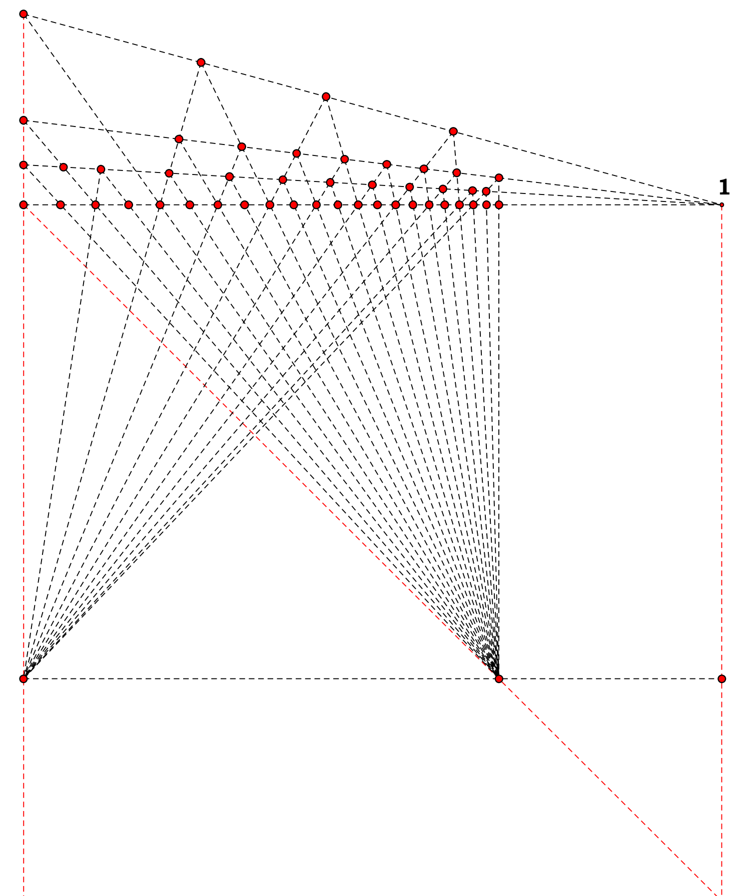
Here I take 3 times 7 to produce the 21st root series. Since these are not adjacent on the loci, one is going to have to make a couple passes with the line as in the following.



Multiplication 3x7 2

Thursday, November 18, 2021

It took three passes, each one adding more roots.



Multiplication 3x7 3

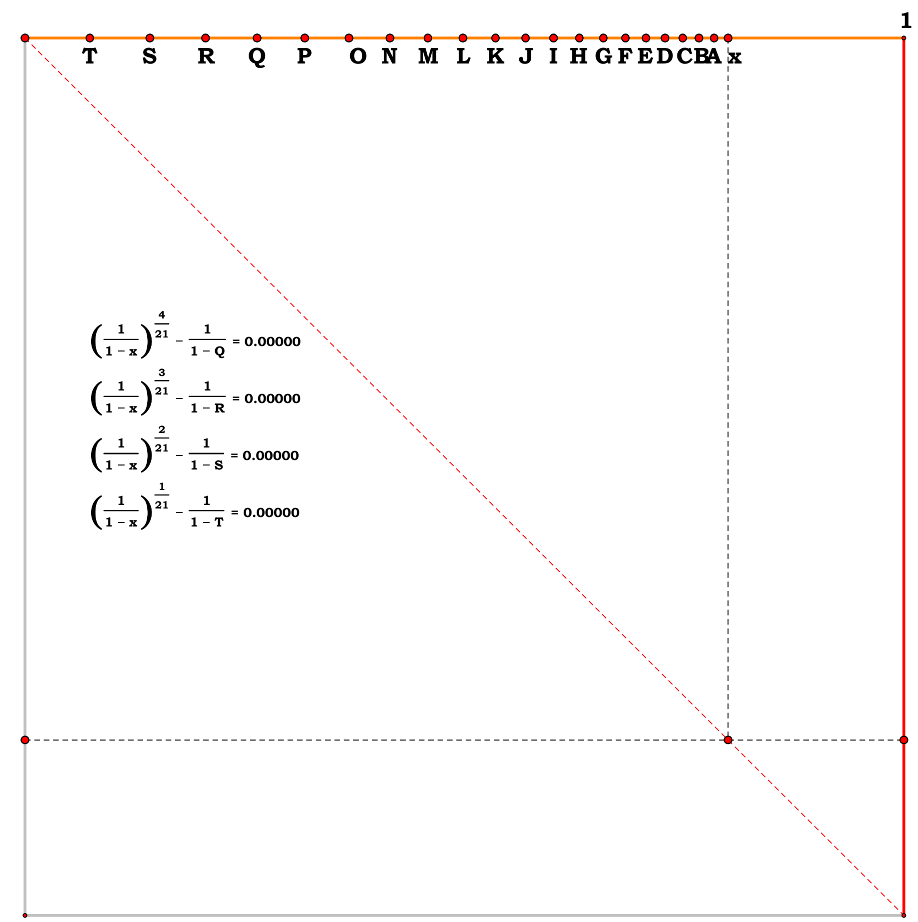
Thursday, November 18, 2021

And there is all of the twenty-first roots of the number 5.

x = 0.80000
A = 0.78407
B = 0.76687
C = 0.74830
D = 0.72825
E = 0.70661
F = 0.68324
G = 0.65800
H = 0.63076
I = 0.60135
J = 0.56960
K = 0.53532
L = 0.49830
M = 0.45834
N = 0.41520
O = 0.36861
P = 0.31832
Q = 0.26403
R = 0.20540
S = 0.14211
T = 0.07378

$$\frac{1}{1-x} = 5.00000$$
$$\frac{1}{1-A} = 4.63112$$
$$\frac{1}{1-B} = 4.28945$$
$$\frac{1}{1-C} = 3.97299$$
$$\frac{1}{1-D} = 3.67987$$
$$\frac{1}{1-E} = 3.40838$$
$$\frac{1}{1-F} = 3.15693$$
$$\frac{1}{1-G} = 2.92402$$
$$\frac{1}{1-H} = 2.70829$$
$$\frac{1}{1-I} = 2.50848$$
$$\frac{1}{1-J} = 2.32342$$
$$\frac{1}{1-K} = 2.15200$$
$$\frac{1}{1-L} = 1.99324$$
$$\frac{1}{1-M} = 1.84618$$
$$\frac{1}{1-N} = 1.70998$$
$$\frac{1}{1-O} = 1.58382$$
$$\frac{1}{1-P} = 1.46697$$
$$\frac{1}{1-Q} = 1.35874$$
$$\frac{1}{1-R} = 1.25850$$
$$\frac{1}{1-S} = 1.16565$$
$$\frac{1}{1-T} = 1.07965$$

$$\left(\frac{1}{1-x}\right)^{\frac{20}{21}} - \frac{1}{1-A} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{19}{21}} - \frac{1}{1-B} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{18}{21}} - \frac{1}{1-C} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{17}{21}} - \frac{1}{1-D} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{16}{21}} - \frac{1}{1-E} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{15}{21}} - \frac{1}{1-F} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{14}{21}} - \frac{1}{1-G} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{13}{21}} - \frac{1}{1-H} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{12}{21}} - \frac{1}{1-I} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{11}{21}} - \frac{1}{1-J} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{10}{21}} - \frac{1}{1-K} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{9}{21}} - \frac{1}{1-L} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{8}{21}} - \frac{1}{1-M} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{7}{21}} - \frac{1}{1-N} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{6}{21}} - \frac{1}{1-O} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{5}{21}} - \frac{1}{1-P} = 0.00000$$



$$\left(\frac{1}{1-x}\right)^{\frac{4}{21}} - \frac{1}{1-Q} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{3}{21}} - \frac{1}{1-R} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{2}{21}} - \frac{1}{1-S} = 0.00000$$
$$\left(\frac{1}{1-x}\right)^{\frac{1}{21}} - \frac{1}{1-T} = 0.00000$$

I found this abandoned project which I was in the process of illustrating with Jabob's Ladder from:

CONTEMPORARY

INTERMEDIATE

ALGEBRA

CHARLES J. MERCHANT

University of Arizona

I wanted to write an illustrated book on Algebra and at this time, believed that I could use such works as the following for a template. But then, after doing the digital conversion of the work, I realized that this pattern may be helpful, as in the following to explain how to manipulate a naming convention, that the entire work does not actually teach algebra. A grammar book teaches one how to apply names by real examples, and not just how to manipulate names out of context, they do not teach children how to use algebra by pairing the perceptible with the intelligible. To anyone of sense, these are mainly empty words. Secondly, this fetish with polynomial equations is no more than taking one arithmetic name and making a puzzle out of it for no apparent purpose; this is not an act of intelligence.

So, I use this part of that project simply as reference material on exponents.

4.6 EXPONENTS

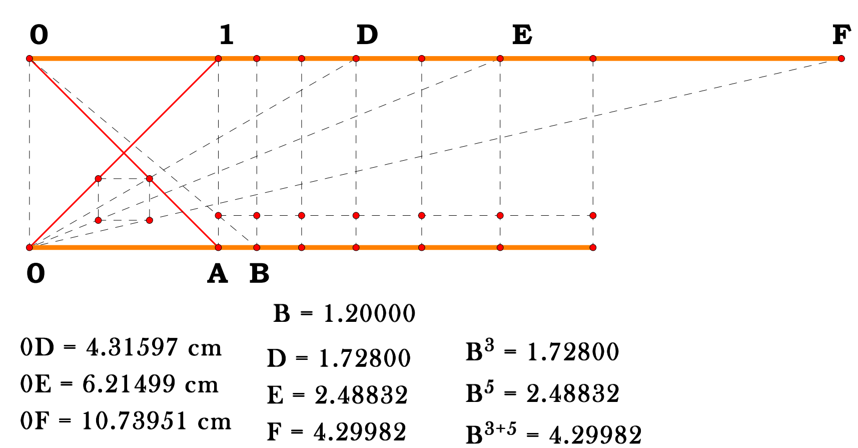
All operations with monomials are greatly facilitated by the use of exponents. The use of exponents not only results in a gain in compactness and legibility, but also greatly facilitates computations involving multiplication and division of monomials. This facility arises as a result of the laws of exponents, which we shall now develop for the case where the exponents involved are positive integers. The extension of the laws of exponents to the case of fractional, negative, and zero exponents will be made in Chapter 7.

THE FIRST LAW OF EXPONENTS

The first law of exponents states:

If *m* and *n* are positive integers and *a* is any number whatever, then

$$a^m \cdot a^n = a^{m+n}$$



This law follows immediately from the definition of positive integral exponents; if m and n are positive integers, then

$$a^m = \overbrace{a \cdot a \cdot a \cdots a}^{m \text{ factors}} \quad \text{and} \quad a^n = \overbrace{a \cdot a \cdot a \cdots a}^{n \text{ factors}}$$

Hence

$$a^m \cdot a^n = \overbrace{a \cdot a \cdot a \cdots a}^{m \text{ factors}} \cdot \overbrace{a \cdot a \cdot a \cdots a}^{n \text{ factors}} = \overbrace{a \cdot a \cdot a \cdots a}^{m+n \text{ factors}} = a^{m+n}$$

THE SECOND LAW OF EXPONENTS

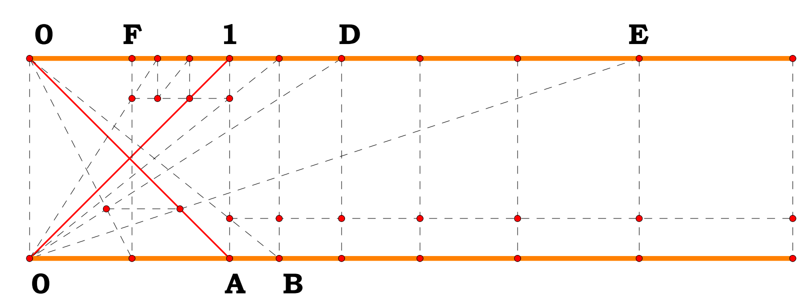
The second law of exponents covers two cases:

1. If m and n are positive integers and $m > n$, then

$$\frac{a^m}{a^n} = \frac{1}{a^{m-n}}, \quad a \neq 0$$

2. If m and n are positive integers, and $m < n$, then

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad a \neq 0$$



B = 1.25000

D = 1.56250
E = 3.05176
F = 0.51200

B^2 = 1.56250
B^5 = 3.05176
B^{-3} = 0.51200

$B^{2\text{-}5}$ = 0.51200
$\frac{1}{B^3}$ = 0.51200

We prove the first case as follows:

$$\frac{a^m}{a^n} = \frac{\overbrace{a \cdot a \cdot a \cdots a}^{m \text{ factors}}}{\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}}$$

Then if $m > n$, we cancel out n factors from the numerator against n in the denominator, and have

$$\frac{a^m}{a^n} = \frac{\overbrace{a \cdot a \cdot a \cdots a}^{m\text{ - }n\text{ factors}}}{\underbrace{a \cdot a \cdot a \cdots a}_{1}}$$

The proof for case 2 is left as an exercise.

THE THIRD LAW OF EXPONENTS

The third law of exponents states: If m and n are positive integers and a is any number whatever, then

$$(a^m)^n = a^{mn}$$

we have

$$(a^m)^n = \underbrace{(\overbrace{a \cdot a \cdot a \cdots a}^{m\text{ factors}}) \cdot (\overbrace{a \cdot a \cdot a \cdots a}^{m\text{ factors}}) \cdots (\overbrace{a \cdot a \cdot a \cdots a}^{m\text{ factors}})}_{n\text{ groups of factors}}$$

$$\overset{mn\text{ factors}}{= a \cdot a \cdot a \cdots a}$$

$$= a^{mn}$$

From the nature of this proof, it is obvious that

$$(a^m)^n = (a^n)^m$$

THE FOURTH LAW OF EXPONENTS

The fourth law of exponents states

$$(ab)^n = a^n b^n \qquad (n \text{ a positive integer})$$

The proof follows immediately from the associative and commutative laws for multiplication.

$$(ab)^n = \underbrace{(a \cdot b) \cdot (a \cdot b) \cdots (a \cdot b)}_{n\text{ parentheses}}$$

$$= \underbrace{a \cdot a \cdot a \cdots a}_{n\text{ factors}} \cdot \underbrace{b \cdot b \cdot b \cdots b}_{n\text{ factors}}$$

$$= a^n \cdot b^n \qquad (n \text{ a positive integer})$$

EXPONENTS, ROOTS AND RADICALS

7.1 ZERO EXPONENTS

The expression a^m, as defined so far, only has meaning when m is a positive integer, since it tells how many times a is used as a factor. a^0, therefore, has no significance at this point, since to say that a number a is used zero times as a factor is meaningless.

We are, therefore, free to define a^0 however we please, as long as we do so in a way that is consistent with the definitions we have already made. We do this by considering the possible meaning of a^0 in relation to other laws of exponents.

The first law of exponents states:

$$\text{I. } a^m = a^{m+n}$$

Let $m = 0$. Then we have

$$a^0 \cdot a^n = a^{0+n} = a^n$$

or

$$a^0 = \frac{a^n}{a^n} = 1, \text{ if } a \neq 0$$

Then consider the two cases of the second law of exponents:

$$\text{II(a). } \frac{a^m}{a^n} = a^{m-n}, m > n, a \neq 0$$

and

$$\text{II(b). } \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad , n > m \; a \neq 0$$

If we let $m = n$, we have

$$\text{II(a). } \frac{a^m}{a^m} = a^0 \qquad \text{or} \qquad 1 = a^0$$

$$\text{II(b). } \frac{a^m}{a^m} = \frac{1}{a^0} \qquad \text{or} \qquad 1 = \frac{1}{a^0}$$

The third law of exponents states:

$$\text{III. } (a^m)^n = a^{mn}$$

Let $m = 0$. Then $(a^0)^n = a^{0 \cdot n} = a^0$. But $1^n = 1$. For all these cases, therefore, it is consistent with the previous statement of the laws of exponents to let $a^0 = 1$. The only thing we must avoid is to let a also equal 0, for then we would be dividing by zero in Law II(a) and II(b). We, therefore, make the following definition.

DEFINITION: $a^0 = 1, a \neq 0$.

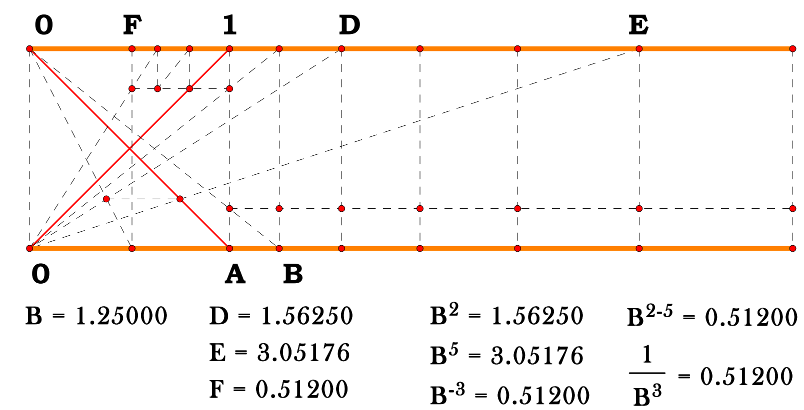
Also, 0^0 is undefined.

7.2 NEGATIVE EXPONENTS

We may proceed to define negative exponents along exactly the same lines as were followed in defining zero exponents in Section 7.1. Consider again Law II, which states:

$$\text{II(a). } \frac{a^m}{a^n} = a^{m-n}, m > n$$

$$\text{II(b). } \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad , n > m$$



Now let us assume that m = 0 and that n is greater than zero. Let us also disregard for the moment the requirement in Law II(a) that m be greater than n. We could then write

$$\frac{a^0}{a^n} = a^{0 - n} = a^{-n} = \frac{1}{a^{n-0}} = \frac{1}{a^n}$$

or, in other words,

$$a^{-n} = \frac{1}{a^n}$$

Thus, 3^{-2} =. $1/3^2$, and so forth.

If we were to define a^{-n} as equal to $1/a^n$, there would be one obvious immediate advantage: It would no longer be necessary to consider two separate cases under Law II. We could simply write

$$\frac{a^m}{a^n} = a^{m-n}$$

without having to consider whether m was greater than n or not. This would be a great convenience. Thus, we would have

$$\frac{3^6}{3^4} = 3^2$$

or, again,

$$\frac{3^4}{3^6} = 3^{-2}$$

and, finally,

$$\frac{3^4}{3^4} = 3^0$$

Thus, only one statement of Law II would be necessary, assuming m and n were both positive. Since we easily show that there is no inconsistency with the other laws of exponents, we therefore make the following definition:

DEFINITION: $a^{-n} = \dfrac{1}{a^n}$, $a \neq 0$.

EXERCISE 7A

Rewrite, using positive exponents.

Example 1: x^{-2}

SOLUTION:

$$x^{-2} = \frac{1}{x^2}$$

Example 2: $(a^{-3})^{-2}$

SOLUTION:

$$(a^{-3})^{-2} = a^{(-3)(-2)} = a^6$$

1. $x^{-2}z^{-1}$
2. a^3b^{-2}
3. $3^{-2}a^{-4}b^{-6}$
4. $(-5)^{-3}a^4b^{-1}$

5. $(-2)^{-1}(-4)^{-2}a^5$

6. $(b^{-1})^{-1}$

7. $(p^{-2})^4(q^{-1})^{-2}$

8. $(x^{-1}y^{-5}z^2)^{-3}$

9. $\dfrac{x^4}{y^{-2}}$ 10. $\dfrac{(-1)^{-1}}{(-5)^{-2}x^{-7}}$

11. $\dfrac{(-5)^{-2}(-5)^{-3}}{x^{-2}y^2}$ 12. $\dfrac{a^{-1}b^{-2}c^{-3}}{x^{-3}y^{-2}z^{-1}}$

13. $\dfrac{1}{(-r)^{-2}(s^{-3})^{-1}t}$ 14. $\dfrac{(-1)^0y^{-2}}{3^0x^2y^{-2}}$

15. $\dfrac{a^5b^{-5}(c^{-5})^{-1}}{2^{-5}}$

Perform the indicated operations. Express your answer as a product of terms, using negative exponents where necessary.

Example 3: $\dfrac{x^2}{y^2}$

SOLUTION:

$$\frac{x^2}{y^2} = x^2y^{-2}$$

16. $\dfrac{a}{b}$ 17. $\dfrac{1}{ab}$

18. $\dfrac{x^2}{y^2z^2}$ 19. $\dfrac{ac}{b^{-1}}$

20. $\dfrac{(xy)^2}{(xy)^3}$ 21. $\dfrac{a^4b^{-2}c^5}{a^5b^{-3}c^4}$

22. $\dfrac{-2p^{-1}q^4}{-p^{-2}q^5r}$ 23, $\left(\dfrac{xy^2}{x^3y}\right)^{-1}$

24 $\left(-\dfrac{a^{-2}b^{-3}}{a^{-4}b^3}\right)^{-2}$ 25. $(-3)^{-2}(-4)^{-1}\dfrac{1}{x^2y^{-3}}$

Simplify and express your answer as a single fraction using only positive exponents.

Example 4: $a^{-1} + b^{-2}$

$$a^{-1} + b^{-2} = \frac{1}{a} + \frac{1}{b^2}$$

$$= \frac{b^2}{ab^2} + \frac{a}{ab^2}$$

$$= \frac{a + b^2}{ab^2}$$

26. $a^{-2} + 4^{-1}$ 27. $3^{-1} + 9^{-1}$

28. $2(-3)^{-2} - 3(-2)^{-1}$ 29. $4 \cdot 5^{-1} - 3 \cdot 5^{-2}$

30. $10^{-3} - 10^{-4}$

31. $a^{-4} - a^{-5}$

32. $2ab^{-1} - a^{-2}b$

33. $(a + a^{-1})^{-1}$

34. $(x^{-2} - x^{-1})^{-1}$

35. $(-2x^{-1})^{-2} - 3x^{-2}$

36. $\left(\dfrac{1}{x^{-1}} + \dfrac{2}{x^{-2}}\right)^{-1}$

37. $\left(\dfrac{a^{-3} + b^{-1}}{a^{-1} \cdot b^{-1}}\right)^{-1}$

7.2 **93**

38. $\left(\frac{a^{-3}}{b^{-2}} - \frac{b^{-2}}{a^{-3}}\right)^{-1}$

39. $-\frac{1}{2} \cdot a^{-3} - (-2)^{-2} b^{-2} - (-1)^{-1} c^{-1}$

40. $-\frac{x^{-2}}{y^{-1}} - \frac{y^{-3}}{z^{-2}} + \frac{z^{-1}}{x^{-3}}$

7.3 ROOTS; RADICALS

We have seen that it is both useful and consistent with the fundamental laws of exponents as stated for positive integral exponents to assign meaning to zero and negative integral exponents. We next inquire whether it might Also be useful and consistent to assign meaning to fractional exponents, that is, exponents of the form m/n, where m and n are integers.

We first define what is meant by the "nth root" of a positive number.

DEFINITION: The nth root of a positive number a is that number which multiplied together n times (that is, taken n times as a factor) equals the given number a.

Example 1: The third root of 8 is 2, since $2 \cdot 2 \cdot 2 = 8$.

Example 2: The fifth root of 7776 is 6, since $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 7776$.

The symbol for the nth root of a is $\sqrt[n]{a}$. In this symbol, n is called the index of the root or the order of the root; the symbol $\sqrt{\ }$ is called the radical sign; the number a is called the radicand, the whole expression $\sqrt[n]{a}$ is called a radical.

The second root of a is called the square root of a, and in this case, the index is normally omitted from the radical. Thus,

$$\sqrt{a} = \sqrt[2]{a}$$

The third root of a number is called its cube root. The fourth, fifth, and higher order roots are called simply the fourth root, fifth root, and so on. In all cases except the square root, the index is always shown.

With these meanings assigned to these symbols, we may then write the following:

$$\left(\sqrt[n]{a}\right)^n = a = a^1$$

which simply states the definition of the nth root symbolically.

From the definition of the nth root of a, it also follows immediately that

$$\sqrt[n]{a^n} = a$$

When a is the nth power of a rational number, say p/q, where p and q are integers, then we can write $a = (p/q)^n$, whence it follows that

$$\sqrt[n]{a}\ , = \sqrt[n]{\left(\frac{p}{q}\right)^n} = \frac{p}{q}$$

by the preceding paragraph. However, when a is not the nth power of a rational number, then $\sqrt[n]{a}$ is not a rational number, and in this case we say that $\sqrt[n]{a}$ is irrational.

7.4 FRACTIONAL EXPONENTS

Now consider Law III, which states

$$(a^{1/m})^n = a^{mn}$$

and suppose that for a^m we write $\sqrt[n]{a}$. Then

$$\left(\sqrt[n]{a}\right)^n = a^{mn} = a^1$$

But if $a^{mn} = a^1$, we must have mn or $m = 1/n$. We, therefore, set $a^m = a^{1/n}$, and write

$$(a^{1/n})^n = a^{(1/n)\cdot n} = a^1 = a$$

Let us tentatively assign the meaning $\sqrt[n]{a}$ to the symbol $a^{1/n}$ and investigate the consequences. As a concrete example, let us write $\sqrt[3]{a} = a^{1/3}$. Then what would be the meaning of $a^{2/3}$? From the first law of exponents,

$$a^{1/3}\cdot a^{1/3} = a^{(1/3)+(1/3)} = a^{2/3}$$

But from the third law of exponents

$$(a^{1/3})^2 = a^{(1/3)\cdot 2} = a^{2/3}$$

These two meanings are, of course, the same; $(a^{1/3})^2 = a^{(1/3)}\cdot a^{(1/3)}$. We see from this example immediately that if m and n are integers, then we may always write

$$(a^{1/n})^m = \underbrace{a^{1/n}\cdot a^{1/n}\cdots a^{1/n}}_{m \text{ factors}} = a^{m/n}$$

and that this is consistent with the meaning of $\left(\sqrt[n]{a}\right)^n$ if we define $a^{1/n}$ as equal to $\sqrt[n]{a}$.

It was observed in the case where m and n were positive integers that

$$(a^m)^n = (a^n)^m = a^{mn}$$

or, in other words, that the commutative law held for multiplication of exponents when these were positive integers. If this commutative law is to hold for fractional exponents, we must have

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$$

We have shown that $a^{m/n}$ is equal to $(a^{1/n})^m$. Let us now show that $(a^{1/n})^m = (a^m)^{1/n}$, where m and n are positive integers.

By the definition of $a^{1/n}$,

$$a = \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{n \text{ factors}}$$

Therefore,

$$a^m = \underbrace{\underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{n \text{ factors}} \cdot \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{n \text{ factors}} \cdots \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{n \text{ factors}}}_{m \text{ groups of factors}}$$

and therefore,

$$a^m = \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{mn \text{ factors}}$$

Hence,

$$(a^m)^{1/n} = \sqrt[n]{a^m} = \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{m \text{ factors}}$$

since

$$[(a^m)^{1/n}]^n = \underbrace{\underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{m \text{ factors}} \cdot \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{m \text{ factors}} \cdots \underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{m \text{ factors}}}_{n \text{ groups of factors}}$$

Therefore,

$$(\underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}_{m \text{ factors}}) = \sqrt[n]{a^m}$$

by the definition of the nth root. But obviously,

$$(\underbrace{a^{1/n} \cdot a^{1/n} \cdots a^{1/n}}) $$

m factors

is equal to $(a^{1/n})^m$. Therefore,

$$(a^m)^{1/n} = (a^{1/n})^m$$

and the commutative law holds here also.

Since none of the foregoing shows any conflict with the first three laws of exponents, we therefore make the following definitions:

DEFINITION: $a^{1/n} = \sqrt[n]{a}$ (a a positive real number).

DEFINITION: $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ (n a positive integer)

$= \sqrt[n]{a^m}$ (a a positive real number, m and n positive integers).

Using the definition of negative exponents already developed, these definitions may be extended immediately to the case of a negative fractional exponent. Thus,

Example 1:

$$a^{-2/3} = 1/a^{2/3}$$

$$= 1/(\sqrt[3]{a})^2$$

$$= 1/\sqrt[3]{a^2}$$

7.5 THE FOURTH LAW OF EXPONENTS FOR FRACTIONAL EXPONENTS

The fourth law of exponents (see Section 4.6) has an important consequence in the case of roots of numbers. The fourth law of exponents states

$$(ab)^n = a^n b^n \quad (n \text{ a positive integer})$$

In the case of the nth root of a number, this law reads

$$\sqrt[n]{ab} = (ab)^{1/n}$$

$$= a^{1/n} b^{1/n}$$

$$= \sqrt[n]{a}\sqrt[n]{b}$$

This result is used in the process of finding roots by factoring, as illustrated in the following examples:

Example 1: Find $\sqrt{7056}$.

SOLUTION: Factoring 7056 completely, we find

$$7056 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 \cdot 7$$

$$= 2^2 \cdot 2^2 \cdot 3^2 \cdot 7^2$$

$$\therefore \sqrt{7056} = \sqrt{2^2 \cdot 2^2 \cdot 3^2 \cdot 7^2}$$

$$= \sqrt{2^2}\sqrt{3^2}\sqrt{7^2}$$

$$= 2 \cdot 2 \cdot 3 \cdot 7$$

$= 84$

The process may be extended to roots of any order. Thus,

Example 2: find $\sqrt[3]{1728}$.

71 97

SOLUTION: Factoring 1728 completely, we have

$$1728 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

$$= 2^3 \cdot 2^3 \cdot 3^3$$

$$\therefore \sqrt[3]{1728} = \sqrt[3]{2^3 \cdot 2^3 \cdot 3^3}$$

$$= \sqrt[3]{2^3} \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{3^3}$$

$$= 2 \cdot 2 \cdot 3$$

$$= 12$$

This process may also be used to amplify a root by bringing factors outside the radical sign. Thus,

Example 3: simplify $\sqrt{99}$ as much as possible.

SOLUTION:

$$\sqrt{99} = \sqrt{3 \cdot 3 \cdot 11} = \sqrt[3]{11}$$

It should be noted strongly that, in general, $\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$ unless either a or $b = 0$. As a matter of fact, there is usually no way to add two or more radicals together. However, on occasion the above process may be used to do so. Thus,

Example 4: add, if possible, $\sqrt{245} + \sqrt{320}$.

SOLUTION:

$$\sqrt{245} + \sqrt{320} = \sqrt{7 \cdot 7 \cdot 5} + \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$$

$$= \sqrt{7^2} \cdot \sqrt{5} + \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{5}$$

$$= 7\sqrt{5} + 8\sqrt{5}$$

$$= 15\sqrt{5}$$

The proof of the fifth law of exponents for fractional exponents is left as an exercise.

EXERCISE 7.1

Rewrite, using fractional exponents.

1. $\sqrt{3}$
2. $\sqrt[3]{5}$
3. $\sqrt[4]{4^3 \cdot x^3}$
4. $\sqrt[7]{5^5 \cdot a^3}$
5. $\sqrt[10]{2^3 \cdot x^9 \cdot y^7}$
6. $\dfrac{1}{\sqrt{7}}$
7. $\dfrac{1}{\sqrt[3]{3^2}}$
8. $\dfrac{1}{\sqrt[5]{2^3 \cdot x^4}}$

9. $\dfrac{1}{\sqrt[7]{3^5 \cdot x^3 \cdot y^3}}$

10. $\dfrac{1}{\sqrt[9]{10 \cdot x \cdot z^2}}$

y

Rewrite as radicals. Show two forms.

11. $5^{3/5}$
12. $2^{7/10}$
13. $x^{3/4}$
14. $y^{2/5}$
15. $2^{2/3} \cdot x^{2/3}$
16. $3^{-2/3}$
17. 4- 3/5
18. , - 7/10
19. $x^{-3/7} \cdot y^{-3/7}$
20. $2^{-9/13} \cdot x^{-9/13} \cdot y^{-9/13}$

Factor completely, and find the root.

21. $\sqrt{324}$
22. $\sqrt{576}$
23. $\sqrt{196}$
24. $\sqrt{225}$
25. $\sqrt{1764}$
26. $\sqrt[3]{1728}$
27. $\sqrt[3]{8000}$
28. $\sqrt[4]{1296}$
29. $\sqrt[5]{7776}$
30. $\sqrt[5]{248{,}832}$

Factor completely, and simplify as much as possible.

31. $\sqrt{80}$
32. $\sqrt{175}$
33. $\sqrt{243}$
34. $\sqrt{7500}$
35. $\sqrt{432}$
36. $\sqrt[3]{81}$
37. $\sqrt[3]{750}$
38. $\sqrt[4]{112}$
39. $\sqrt[4]{405}$
40. $\sqrt[5]{160}$

Simplify as much as possible, and add.

41. $\sqrt{45} + \sqrt{20}$
42. $\sqrt{98} + \sqrt{50}$
43. $\sqrt{363} + \sqrt{48}$
44. $\sqrt{288} - \sqrt{50}$
45. $\sqrt{125} - \sqrt{45} + \sqrt{20}$
46. $\sqrt[3]{16} + \sqrt[3]{54}$
47. $\sqrt[3]{375} - \sqrt[3]{81}$
48. $\sqrt[3]{128} - \sqrt[3]{250}$
49. $\sqrt[3]{256} - \sqrt[3]{500} + \sqrt[3]{32}$
50. $\sqrt[4]{162} + \sqrt[4]{1250} - \sqrt[4]{32}$
51. Prove

$\dfrac{a^{m/n}}{b^{m/n}} = \left(\dfrac{a}{b}\right)^{m/n}$ (*m* and *n* positive integers, *a* and *b* positive real numbers)

7.6 ROOTS OF NEGATIVE NUMBERS: PRINCIPAL ROOT

We have shown that if a is a positive number, and m and n are integers, it is consistent with laws of exponents as previously stated to make the following definitions:

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

$$= (a^m)^{1/n} = \sqrt[n]{a^m}$$

In the case where n is an even integer, however, the expression $\sqrt[n]{a}$ is ambiguous. For $(2)^2 = 4$, and $(-2)^2 = 4$, and hence by the definition of the square root of a number, both 2 and -2 are square roots of 4. Similarly, both 3 and - 3 are sixth roots of 729, since $(3)^6 = 729$, and $(-3)^6 = 729$.

We therefore define the principal nth root, in those cases where the number has both a positive and a negative nth root. This situation occurs only when the given radicand is positive, and n is even.

In these cases, when we write $\sqrt[n]{a}$ we *always* mean the *positive* nth root; when we wish to specify the negative nth root, we *always* write $-\sqrt[n]{a}$. Thus, $\sqrt{4} = 2$, and $-\sqrt{4} = -2$; $\sqrt[6]{729} = 3$, and $-\sqrt[6]{729} = -3$.

Negative numbers do not have even real roots. $\sqrt{-4} \neq 2$, since $(2)^2 = 4$; similarly, $\sqrt{-4} \neq -2$, since $(-2)^2 = 4$. Thus, there is no real number which multiplied by itself yields - 4. It is for this reason that it was stated (Section 3.3) that the real number system is not closed with respect to the taking of roots; real numbers whose square roots are not real numbers exist. In order to take even roots of negative numbers, we must introduce a new kind of number, the "imaginary" number; this topic will be dealt with in the next section.

In the case where n is odd, any real number, positive or negative, has one and only one real nth root. Thus, $\sqrt[3]{27} = 3$, and $\sqrt[3]{-27} = -3$; $\sqrt[5]{32} = 2$ and $\sqrt[5]{-32} = -2$, and so forth. In the case where n is odd, we call this real root the principal nth root of the given number.

7.7 IMAGINARY NUMBERS: COMPLEX NUMBERS

In order to take even roots of negative numbers, a further extension of the number system is necessary. No real number can be the square root of a negative number. We therefore define another kind of number, called the imaginary unit i. This number we define by the relationship

$$i^2 = -1$$

We may then write

$$\sqrt{-1} = i$$

since $i \cdot i = -1$

Then if we are given any negative number, we may find its square root in terms of i. Thus,

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

Any multiple of i, such as $3i$, $-2i$, and so forth, is called an imaginary number. The sum of a real number and an imaginary number, such as $2 + 3i$, is called a complex number. Any complex number may be written in the form $a + bi$, where a and b are real numbers.

The sum of two complex numbers is defined as the sums of their real and complex parts. Thus,

DEFINITION:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

The product of two complex numbers is found by using the distributive law. Thus,

$$(a_1 + b_1 i) \cdot (a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2) + (a_2 b_1)i$$

It is not necessary to remember this formula; in any actual problem, the work is performed simply as though two binomials were being multiplied together, and terms are then collected.

Example 1: Multiply $(2 + 3i)$ by $(-4 + i)$.

SOLUTION:

$$(2 + 3i) \cdot (-4 + i) = -8 + 2i - 12i + 3i^2$$

$$= (-8 - 3) + (2 - 12)i$$

$$= -11 - 10i$$

As in the case of real numbers, subtraction is defined as the inverse of addition, and division is defined as the inverse of multiplication. Thus,

DEFINITION:

$$(a_1 + b_1 i) - (a_2 + b_2 i)$$

$$= (a_1 + b_1 i) + [-(a_2 + b_2 i)]$$

$$= a_1 + b_1 i - a_2 - b_2 i$$

$$= (a_1 - a_2) + (b_1 - b_2)i$$

Similarly,

DEFINITION:

$$(a_1 + b_1 i) \div (a_2 + b_2 i) = (a_1 + b_1 i) \cdot \left(\frac{1}{a_2 + b_2 i}\right)$$

The actual "mechanics" of division of complex numbers will be dealt with later.

Two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal. Thus, if $a + bi = c + di$, then $a = c$ and $b = d$.

A complex number equals zero if, and only if, both its real part and its imaginary part equal zero. Thus, if $a + bi = 0$, then $a = 0$ and $b = 0$.

The complex numbers are not "ordered." It is not correct to say that one complex number is greater than another complex number.

7.8 THE COMPLEX NUMBER SYSTEM

Both the real numbers and the imaginary numbers are proper subsets of the set of all complex numbers. The extension of the number system to embrace the, complex numbers is the final extension necessary for the purposes of elementary algebra. It may be shown that any complex number has an *n*th root which is itself a complex number. The set of complex numbers, which includes the real numbers, is thus "closed" with respect to the operation of root taking, and the final defect, as far as the elementary operations are concerned, is thus removed.

Not only does every complex number have a complex *n*th root, but the following extremely important and basic theorem may be shown to be true:

THEOREM 7.8.1: Every complex number has *n* distinct *n*th roots.

In this theorem, the word "distinct" means "different from the other *n*th roots of this number."

The proof of this theorem is outside the scope of this book. The process of finding the *n* distinct *n*th roots of a complex number is covered by a famous theorem, known as **DeMoivre's theorem**, but this requires the use of trigonometry, and will not be considered here. Certain special cases will, however, be dealt with later.

We may always *verify* that a given complex number is the *n*th root of another given complex number, by the simple process of multiplication.

Example 1: Verify that $\sqrt[3]{-2+2i} = 1 + i$.

SOLUTION:

$$(1+i)^3 = (1+i)\cdot[(1+i)\cdot(1+i)]$$
$$= (1+i)\,[1+i+i-1]$$
$$= (1+i)\,(2i)$$
$$= -2+2i$$
$$\therefore\ 1 + i = \sqrt[3]{-2+2i}$$

EXERCISE 7.3

Express the following complex numbers in terms of *i*. Simplify as much as possible.

Example 1: $\sqrt{-7}$

SOLUTION:

$$\sqrt{-7} = \sqrt{7}i$$

Example 2: $4 - \sqrt{-12}$

SOLUTION:

$$4 - \sqrt{-12} = 4 - \sqrt{12}i = 4 - 2\sqrt{3}i$$

1. $\sqrt{-1}$ 2. $\sqrt{-25}$
3. $\sqrt{-5}$ 4. $\sqrt{-18}$
5. $\sqrt{-50}$ 6. $1+\sqrt{-1}$
7. $2+\sqrt{-2}$ 8. $3+\sqrt{-18}$
9, $-5-2\sqrt{-8}$ 10. $3-5\sqrt{-7}$

Add and subtract the following complex numbers as indicated.

11. $(2+3i)+(1-i)$ 12. $(4-i)+(2+i)$
13. $(-2-i)+(2-i)$ 14. $(17-5i)+(-3-i)$
15. $(12+i)+(-10-i)$ 16. $(3-i)-(1+i)$
17. $(14-2i)-(7-i)$ 18. $(-4+i)-(4+i)$
19. $(10+3i)-(-10-4i)$ 20. $(11-11i)-(-11-11i)$

Multiply the following complex numbers together.

21. $(1+i)\cdot(1+2i)$ 22. $(3-i)\cdot(2+i)$
23. $(5+2i)\cdot(2-i)$ 24. $(1+i)\cdot(1-i)$
25. $(5-2i)\cdot(5+2i)$ 26. $(3+2i)\cdot(3-2i)$
27. $(1+2i)\cdot(1+3i)$ 28. $(-2+i)\cdot(-2-i)$
29. $(-5+7i)\cdot(-5-7i)$ 30. $(4+3i)\cdot(1-i)$

Verify the following.

31. $\sqrt{-2i}=1-i$ 32. $\sqrt{3+4i}=2+i$
33. $\sqrt{-7+24i}=3+4i$ 34. $\sqrt{5-12i}=3-2i$
35. $\sqrt{45-28i}=7-2i$ 36. $\sqrt{120+22i}=11+i$
37. $\sqrt[3]{-11-2i}=1+2i$ 38. $\sqrt[3]{2-11i}=2-i$
39. $\sqrt[4]{-4}=1+i$ 40. $\sqrt[5]{-4+4i}=1-i$

7.9 MANIPULATION OF RADICALS BY MEANS OF FRACTIONAL EXPONENTS

The use of fractional exponents makes it possible to change the form of radicals quickly and easily. Such changes, which are often necessary or desirable for purposes of algebraic manipulation, include the following:

1. changing the index of a radical
2. introduction of factors under the radical sign
3. removal of factors from a radical
4. reduction of a radical to simplest form
5. rationalizing the denominator of a radical.

Each of these procedures will be illustrated by means of appropriate examples.

Example 1: Rewrite $\sqrt{3}$ as a radical of index 6.

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SOLUTION:

$$\sqrt{3} = 3^{1/2} = 3^{3/6} = (3^3)^{1/6} = \sqrt[6]{3^3} = \sqrt[6]{27}$$

Example 2: Rewrite $\sqrt[10]{16}$ as a radical of lowest possible order.
SOLUTION:

$$\sqrt[10]{16} = \sqrt[10]{2^4} = 2^{4/10} = 2^{2/5} = (2^2)^{1/5} = \sqrt[5]{4}$$

Example 3: Rewrite $\sqrt{5}$ and $\sqrt[3]{7}$ as radicals of the same order.

SOLUTION: $\sqrt{5} = 5^{1/2}$, and $\sqrt[3]{7} = 7^{1/3}$. The least common multiple of 2 and 3 is 6. We, therefore, write $5^{1/2} = 5^{3/6} = 6, 125$, and $7^{1/3} = 7^{2/6} = \sqrt[6]{49}$, which are of the same order.

Example 4: Rewrite $\sqrt[4]{5a^2b^3}$ and $\sqrt[10]{8a^5b}$ as radicals of the same order.

SOLUTION: $\sqrt[4]{5a^2b^3} = (5a^2b^3)^{1/4}$, and $\sqrt[10]{8a^5b} = (8a^5b)^{1/10}$. The least common multiple of 4 and 10 is 20. We therefore write $(5a^2b^3)^{1/4} = (5a^2b^3)^{5/20} = [(5a^2b^3)^5]^{1/20} = \sqrt[20]{5^5a^{10}b^{15}} = \sqrt[20]{3125a^{10}b^{15}}$, and $(8a^5b^3)^{1/10} = (8a^5b^3)^{2/20} = [(8a^5b)^2]^{1/20} = \sqrt[20]{8^2a^{10}b^2} = \sqrt[20]{64a^{10}b^5}$.

Example 5: Rewrite $2\sqrt[3]{3}$ as a radical in which the coefficient outside the radical is 1.
SOLUTION:

$2\sqrt[3]{3} = 2(3)^{1/3} = 2^{3/3} \cdot 3^{1/3} = (2^3 \cdot 3^1)^{1/3} = (24)^{1/3} = \sqrt[3]{24}$

Example 6: Rewrite $12a^3\sqrt[5]{6ab^2}$ as a radical in which the coefficient outside the radical is $2a$.
SOLUTION:

$$12a^3\sqrt[5]{6ab^2} = 2a \cdot 6a^2(6ab^2)^{1/5}$$
$$= 2a(6a^2)^{5/5}(6ab^2)^{1/5} = 2a[(6a^2)^5 \cdot (6ab^2)]^{1/5}$$
$$= 2a[6^5 \cdot a^{10} \cdot 6 \cdot a \cdot b^2]^{1/5}$$
$$= 2a[6^6a^{11}b^2]^{1/5}$$
$$= 2a\sqrt[5]{46{,}656a^{11}b^2}$$

Example 7: Remove all possible factors from under the radical sign in
$\sqrt[3]{256x^4y^8z^6}$

SOLUTION:

$$\sqrt[3]{256x^4y^8z^6} = (4^4x^4y^8z^6)^{1/3} = 4^{4/3}x^{4/3}y^{8/3}z^{6/3}$$

$$= 4^{1+(l/3)} \cdot x^{1+(1/3)} \cdot y^{2+(2/3)}) \cdot z^2$$

$$= 4 \cdot 4^{1/3} \cdot x \cdot x^{1/3} \cdot y^2 \cdot y^{2/3} \cdot z^2$$

$$= 4 \cdot x \cdot y^2 \cdot z^2 \cdot 41/3 \,.\, x^{1/3} \cdot y^{2/3}$$

$$= 4xy^2z^2(4xy^2)^{1/3} = 4xy^2z^2\sqrt[3]{4xy^2}$$

The technique exhibited in Example 7 is seen to consist of (1) writing the radical in terms of fractional exponents, (2) writing any exponent which is an improper fraction (that is, a fraction in which the numerator is greater than or equal to the denominator) as the sum of an integer and a proper fraction, (3) grouping the terms with integral exponents in one group and the terms with fractional exponents in another group, and (4) rewriting the terms with fractional exponents as a radical.

When all possible factors have been brought out from under the radical sign, as in Example 7, and the resultant radical, if any, rewritten as a radical of lowest possible order, the result is said to be a radical reduced to simplest form.

Example 8: Reduce to simplest form $\sqrt[4]{64a^{10}b^2}$.

SOLUTION:

$$\sqrt[4]{64a^{10}b^2} = (2^6a^{10}b^2)^{1/4} = 2^{6/4} \cdot a^{10/14} \cdot b^{2/4}$$

$$= 2^{+1+(2/4)} \cdot a^{2+(2/4)} \cdot b^{2/4} = 2 \cdot a^2 \cdot 2^{2/4} \cdot a^{2/4} \cdot b^{2/4}$$

$$= 2 \cdot a^2 \cdot 2^{1/2} \cdot a^{1/2} \cdot b^{1/2} = 2a^2(2ab)^{1/2} = 2a^2\sqrt{2ab}$$

In the case of radicals in which the radicand is a fraction, it is often desirable to rewrite the radical as a fraction in which the only radical which occurs is in the numerator. This process is called rationalizing the denominator. Normally, the result is reduced to simplest form.

Example 9: Rationalize the denominator of $\sqrt[3]{5b/2a^2}$.

SOLUTION: By the fifth law of exponents,

$$\sqrt[3]{\frac{5b}{2a^2}} = \frac{\sqrt[3]{5b}}{\sqrt[3]{2a^2}} = \frac{(5b)^{1/3}}{(2a^2)^{1/3}} = \frac{5^{1/3}b^{1/3}}{2^{1/3}a^{2/3}}$$

Now in order to free the denominator of terms with fractional exponents, each term is multiplied by the term of lowest degree which will make its exponent an integer. Thus, $2^{1/3}$ must be multiplied by $2^{2/3}$, and $a^{2/3}$ must be multiplied by $a^{1/3}$. Thus, $2^{1/3} \cdot a^{2/3} \cdot a^{2/3} \cdot a^{1/3} = 2^{(1/3)+(2/3)} \cdot a^{(2/3)+(1/3)} = 2^1 \cdot a^1 = 2a$. To keep the value of the expression unchanged, the numerator must be multiplied by the same factors. The work is thus written

$$\frac{5^{1/3}\cdot b^{1/3}}{2^{1/3}\cdot a^{2/3}} = \frac{5^{1/3}\cdot b^{1/3}}{2^{1/3}\cdot a^{2/3}}\cdot\frac{2^{2/3}\cdot a^{1/3}}{2^{2/3}\cdot a^{1/3}}$$

$$= \frac{2^{2/3}\cdot 5^{1/3}\cdot a^{1/3}\cdot b^{1/3}}{2a}$$

$$= \frac{\sqrt[3]{20ab}}{2a}$$

in which the denominator is rational.

EXERCISE 7.4

Rewrite the given radical as a radical of the new given order.

1. $\sqrt{2}$; index 6
2. $\sqrt{25}$; index 4
3. $\sqrt{3}$; index 8
4. $\sqrt[3]{3}$; index 9
5. $\sqrt[4]{2}$; index 12
6. $\sqrt[5]{3x}$; index 30
7. $\sqrt[6]{49a^2}$; index 3
8. $\sqrt[4]{25y^4}$; index 2
9. $\sqrt[6]{8x^3}$; index 2
10. $\sqrt[12]{16a^8}$; index 3

Rewrite the following radicals as radicals of the lowest possible order.

11. $\sqrt[6]{8}$
12. $\sqrt[4]{25}$
13. $\sqrt[10]{16}$
14. $\sqrt[12]{81}$
15. $\sqrt[8]{64}$
16. $\sqrt[9]{125x^6}$
17. $\sqrt[6]{64x^3}$
18. $\sqrt[4]{256a^2b^4}$
19. $\sqrt[9]{8x^6y^6}$
20. $\sqrt[10]{81x^8y^6}$

Rewrite the following radicals as radicals of the same order.

21. $\sqrt{5}$ and $\sqrt[3]{7}$
22. $\sqrt{2}$ and $\sqrt[4]{2}$

21 $\sqrt[3]{3}$ and $\sqrt[5]{2}$

24. $\sqrt[4]{2}$ and $\sqrt[6]{7}$
25. $\sqrt[3]{4}$ and $\sqrt[6]{8}$
26. $\sqrt[3]{2x^2}$ and $\sqrt[5]{5x^3}$
27. $\sqrt[4]{7x^5}$ and $\sqrt[8]{5y^4}$
28. $\sqrt{2x}$ and $\sqrt[3]{3y^2}$
29. $\sqrt{x^3}$, $\sqrt[3]{2y}$, and $\sqrt[4]{3z}$
30. $\sqrt[3]{2x^2}$, $\sqrt[8]{4ab}$, and $\sqrt[12]{5xy^2}$

Rewrite the following radicals as radicals in which the coefficient outside the radical sign is 1.

31. $\sqrt[2]{2}$
32. $\sqrt[5]{3}$

33. $\sqrt[4]{5}$ 34. $\sqrt[5]{11}$

35. $\sqrt[8]{2}$ 36. $\sqrt[x]{5}$

37. $a \cdot b^2\sqrt{2a}$ 38. $3x\sqrt[3]{3xy}$

39. $2a^2b\sqrt[3]{2ab^2}$ 40. $5xy^2z^3\sqrt[4]{x^3y^2z}$

Reduce the following radicals to simplest form.

41. $\sqrt{8}$ 42. $\sqrt{125}$

43. $\sqrt[3]{24}$ 44. $\sqrt[4]{243}$

45. $\sqrt[5]{64}$ 46. $\sqrt[3]{16x^4}$

47. $\sqrt[3]{243x^5y^5}$ 48. $\sqrt[4]{64x^6y^8}$

49. $\sqrt[6]{256x^8y^{10}}$ 50. $\sqrt[6]{6561x^{14}z^{20}}$

Rationalize the denominators of the following radicals.

51. $\sqrt{\frac{3}{5}}$ 52. , \f (4, 7)

53. $\sqrt{\frac{1}{3}}$ 54. $\sqrt[3]{\frac{2}{5}}$

55. $\sqrt[4]{\frac{1}{2}}$ 56. $\sqrt{\frac{2}{3x}}$

57. $\sqrt{\frac{3x}{5y}}$ 58. $\sqrt[3]{\frac{7}{3ab}}$

59. $\sqrt[3]{\frac{2a}{7b^2}}$ 60. $\sqrt[3]{\frac{4ab}{5c^2}}$

7.10 MULTIPLICATION OF RADICALS

The fourth law of exponents makes it possible to multiply radicals of the same order. Thus,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b} \qquad \text{(fourth law of exponents)}$$

The resulting expressions are then, ordinarily, reduced to simplest form.

Example 1: Find $\sqrt[3]{ab^2} \cdot \sqrt[3]{12a^5b^2}$ and reduce to simplest form.

SOLUTION:

$$\sqrt[3]{ab^2} \cdot \sqrt[3]{12a^5b^2} = \sqrt[3]{(2ab^2) \cdot (12a^5b^2)}$$

$$= \sqrt[3]{24a^6b^4} = \sqrt[3]{2^3 \cdot 3 \cdot a^6 \cdot b^4}$$

$$= 2a^2b\sqrt[3]{3b}$$

Because of the convention that $\sqrt[n]{a}$ shall always be taken as positive when a is positive and n is even, the mechanical application of the fourth law of exponents can lead to an incorrect result when n = 2 and both a and b represent negative numbers. This case requires special treatment, as shown in the following example,

Example 2: Find $\sqrt{-2} \cdot \sqrt{-6}$.

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SOLUTION: The mechanical application of the fourth law of exponents here would yield

$$\sqrt{-2} \cdot \sqrt{-6} = \sqrt{(-2) \cdot (-6)} = \sqrt{12} = \sqrt{3 \cdot 2^2} = \sqrt[2]{3}$$

This answer is not correct. The correct approach to this problem consists in first expressing $\sqrt{-2}$ and $\sqrt{-6}$ in terms of i as follows:

$$\sqrt{-2} \cdot \sqrt{-6} = \sqrt{2}i \cdot \sqrt{6}i = \sqrt{2} \cdot \sqrt{6} \cdot i^2 = \sqrt{12}\,(-1) = -\sqrt{12} = -\sqrt[2]{3}$$

We therefore have the following rule:

When multiplying together two or more square roots of negative quantities, express each of these first in terms of i, before performing the actual multiplication.

The cases involving the multiplication of even-ordered radicals where the index is greater than 2 involve DeMoivre's theorem and will not be considered here.

When multiplying together two radicals of (the same) odd order, no precautions are necessary. Thus, $\sqrt[3]{-8} \cdot \sqrt[3]{-27} = (-2) \cdot (-3) = 6$, and $\sqrt{-8} \cdot \sqrt{-27} = \sqrt[3]{(-8) \cdot (-27)} = \sqrt{216} = 6$, and either procedure yields the same result.

When multiplying together two radicals of different order, the fourth law of exponents does not apply. It is first necessary to rewrite the two radicals as radicals of the same order, as shown in Examples 3 and 4 in Section 7.9. The fourth law of exponents will then apply, and the work can proceed as above.

Example 3: Find $\sqrt{2} \cdot \sqrt[3]{4}$.

SOLUTION: $\sqrt{2} = 2^{1/2}$ and $\sqrt[3]{4} = (2^2)^{1/3} = 2^{1/3}$. The least common denominator of $\frac{1}{2}$ and $\frac{2}{3}$ is 6. We, therefore, write

$$2^{1/2} \cdot 2^{1/3} = 2^{3/6} \cdot 2^{4/6} = (2^3 \cdot 2^4)^{1/6} = \sqrt[6]{2^7} = 2\sqrt[6]{2}$$

(There is an alternative method available here: Since the base of both exponents is the same, we can use the first law of exponents and write

$$2^{1/2} \cdot 2^{1/3} = 2^{(1/2) + (2/3)} = 2^{7/6} = 2 \cdot 2^{1/6} = \sqrt[6]{2^7} = 2\sqrt[6]{2}$$

It should be noted strongly that this latter alternative is only available when the base of both exponents is the same.)

Example 4: Find $\sqrt[3]{9a^2b} \cdot \sqrt[4]{18a^3b^3}$.

SOLUTION:

$\sqrt[3]{9a^2b} = (3^2 \cdot a^2 \cdot b)^{1/3} = 3^{2/3} \cdot a^{2/3} \cdot b^{1/3}$

$\sqrt[4]{18a^3b^3} = (2 \cdot 3^3 \cdot a^3 \cdot b^3)^{1/4} = 2^{1/4} \cdot 3^{2/4} \cdot a^{3/4} \cdot b^{3/4}$

The least common denominator of $\frac{1}{3}$ and $\frac{1}{4}$ is 12. We write

$$\begin{aligned}(3^2a^2b)^{1/3} \cdot (2 - 3^2 \cdot a^3b^3)^{1/4} &= 3^{8/12} \cdot a^{8/12}b^{4/12} \cdot 2^{3/12} \cdot 3^{6/12} \cdot a^{9/12} \cdot b^{9/12} \\ &= 2^{3/12} \cdot 3^{14/12} \cdot a^{17/12} \cdot b^{13/12} \\ &= 2^{3/2}\, 3^{1 + (2/12)} \,.\, a^{1 + (5/122)} \cdot b^{1 + (1/12)} \\ &= 3 \cdot a \cdot b \cdot 2^{3/12} \cdot 3^{2/12} \cdot a^{5/12} \cdot b^{4/12} \\ &= 3ab\sqrt[12]{72a^5b}\end{aligned}$$

DIVISION OF RADICALS

The fifth law of exponents enables us to divide two radicals of the same order. Thus,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad \text{(fifth law of exponents)}$$

Precautions must be taken here, as in multiplication, when $n = 2$, and when one or the other of the radicands is negative. As in the case of multiplication, however, correct results are always obtained by first expressing the radicals in terms of i. As with multiplication, the case where one or both of the radicands is negative, and where n is even and greater than 2, will not be considered here. When both radicals are of the same order, and n is odd, no difficulties are encountered. As in multiplication, when radicals of different order are to be divided, they must first be changed to radicals of the same order before the fifth law of exponents applies.

Example 1: Find $\sqrt{10} / \sqrt{2}$.

SOLUTION:

$$\frac{\sqrt{10}}{\sqrt{2}} = \sqrt{\frac{10}{2}} = \sqrt{5}$$

Example 2: Find $\sqrt[3]{-51} / \sqrt[3]{3}$.

SOLUTION:

$$\frac{\sqrt[3]{-51}}{\sqrt[3]{3}} = \sqrt[3]{\frac{-51}{3}} = \sqrt[3]{-27} = -3$$

Example 3: Find $\sqrt{-125} / \sqrt{-5}$.

SOLUTION:

$$\frac{\sqrt{-125}}{\sqrt{-5}} = \frac{\sqrt{125i}}{\sqrt{5i}} = \sqrt{\frac{125}{5} \cdot \frac{i}{i}} = \sqrt{25} = 5$$

Example 4: Find $\sqrt{-27}\ /\ \sqrt{3}$.
SOLUTION:

$$\frac{\sqrt{-27}}{\sqrt{3}} = \frac{\sqrt{27}i}{\sqrt{3}} = \sqrt{\frac{27}{3}}\,i = \sqrt{9}\,i = 3i$$

Example 5: Find $\sqrt{27}$ / , $\sqrt[-3]{-3}$.
SOLUTION:

$$\frac{\sqrt{27}}{\sqrt{-3}} = \frac{\sqrt{27}}{\sqrt{3}i} = \frac{\sqrt{27}}{\sqrt{3}} \cdot \frac{1}{i} = \sqrt{\frac{27}{3}} \cdot \frac{1}{i} \cdot \frac{i}{i}$$

$$= 3 \cdot \frac{i}{i^2} = \frac{3i}{-1} = -3i$$

A rule could easily be stated covering the three cases illustrated in Examples 3, 4, and 5, but the simplest rule, covering not only these cases but all cases arising in multiplication of radicals as well is the following: *Whenever dealing with the square roots of negative quantities, always rewrite these radicals first in terms of i.*

This rule is foolproof and completely general. On occasion, it involves a slight amount of unnecessary labor, but this is a small price to pay for avoiding all mistakes.

EXERCISE 7.5

Multiply the following radicals together, and reduce the answer to simplest form. (Make sure the radicals are of the same degree before performing the multiplication.)

1. $\sqrt{6} \cdot \sqrt{8}$
2. $\sqrt{15} \cdot \sqrt{30}$
3. $\sqrt[3]{16} \cdot \sqrt[3]{12}$
4. $\sqrt[5]{48} \cdot \sqrt[5]{162}$
5. $\sqrt{-3} \cdot \sqrt{12}$
6. $\sqrt{5} \cdot \sqrt{-125}$
7. $\sqrt{-8} \cdot \sqrt{-6}$
8. $\sqrt{-35} \cdot \sqrt{-14}$
9. $\sqrt[3]{-10} \cdot \sqrt[3]{-20}$
10. $\sqrt[5]{-27} \cdot \sqrt[5]{-45}$
11. $\sqrt{8} \cdot \sqrt[3]{20}$
12. $\sqrt{18} \cdot \sqrt[3]{54}$
13. $\sqrt[3]{27} \cdot \sqrt[4]{48}$
14. $\sqrt[3]{16} \cdot \sqrt[5]{486}$
15. $\sqrt{12ab^3} \cdot \sqrt{24a^5b^2}$
16. $\sqrt[3]{16x^2y^4} \cdot \sqrt[3]{28xy^2}$
17. $\sqrt{2ax^3} \cdot \sqrt[3]{16a^2x}$
18. $\sqrt{18bc} \cdot \sqrt[3]{12b^2c^2}$
19. $\sqrt{-6xy} \cdot \sqrt{-24x^3y}$
20. $\sqrt{-18a} \cdot \sqrt{22b^2c}$

Divide the following radicals and rationalize the denominator of the quotient.

21. $\sqrt{3} \div \sqrt{5}$
22. $\sqrt{48} \div \sqrt{6}$
23. $\sqrt[3]{81} \div \sqrt[3]{6}$
24. $\sqrt[3]{250} \div \sqrt[3]{18}$

25. $\sqrt{-10} \div \sqrt{-15}$ 26. $\sqrt{-30} \div \sqrt{-14}$

27. $\sqrt{-20} \div \sqrt{15}$ 28. $\sqrt{-242} \div \sqrt{11}$

29. $\sqrt{40} \div \sqrt{-12}$ 30. $\sqrt{98} \div \sqrt{-6}$

31. $\sqrt{8a^2b} \div \sqrt[3]{10ab^2}$ 32. $\sqrt[3]{35x^4y} \div \sqrt{14xy^2}$

33. $\sqrt[5]{3a^4b^2} \div \sqrt{6ab}$ 34. $\sqrt[4]{20xz^2} \div \sqrt{5xz}$

35. $\sqrt[4]{4a^2b^3} \div \sqrt[3]{6ab^2}$ 36. $\sqrt[3]{a^2b} \div \sqrt[5]{2ab}$

37. $\sqrt[6]{120y^4z^2} \div \sqrt[3]{10y^2z}$ 38. $\sqrt[4]{8xz^3} \div \sqrt[3]{2z^4}$

39. $\sqrt[3]{20r^2s^2} \div \sqrt[4]{5r^3s^3}$ 40. $\sqrt[4]{10pq} \div \sqrt[3]{5pq}$

7.12 MANIPULATION OF EXPRESSIONS INVOLVING THE ALGEBRAIC SUM OF TWO OR MORE RADICALS

In expressions involving the algebraic sum of two or more radicals, the invariable first step should consists of reducing these radicals to simplest form. In the first place, it is sometimes found that the radicals may then actually be added to obtain a single term, as was shown in Section 7.5, Example 7. In the second place, even when this is not possible, it considerably simplifies all subsequent work.

The product of two expressions, one or both of which consists of the sum of two or more radicals, is found using the distributive law.

Example 1: Find $\sqrt{2}\,(\sqrt[3]{3} + \sqrt[4]{5})$.

SOLUTION:

$$2\sqrt{2}\,(\sqrt[3]{3} + \sqrt[4]{5}) = \sqrt{2} \cdot \sqrt[3]{3} + \sqrt{2} \cdot \sqrt[4]{5}$$
$$= 2^{1/2} \cdot 3^{1/3} + 2^{1/2} \cdot 5^{1/4}$$
$$= 2^{3/6} \cdot 3^{2/6} + 2^{1/4} \cdot 5^{1/4}$$
$$= (2^3 \cdot 3^2)^{1/6} + (2^2 \cdot 5^1)^{1/4}$$
$$= \sqrt[6]{72}\ \ \sqrt[4]{20}$$

Both these radicals are in simplest form.

Example 2: Find $(\sqrt{8} + \sqrt{75}) \cdot (\sqrt{18} - \sqrt{24})$.

SOLUTION:

$$(\sqrt{8} + \sqrt{75}) \cdot (\sqrt{18} - \sqrt{24}) = (2\sqrt{2} + 5\sqrt{3}) \cdot (3\sqrt{2} - 2\sqrt{6})$$
$$= 6 \cdot 2 - 4\sqrt{12} + 15\sqrt{6} - 10\sqrt{18}$$
$$= 12 - 8\sqrt{3} + 15\sqrt{6} - 30\sqrt{2}$$

The radicals are in simplest form.

Given any expression of the form $a + b$, the expression $a - b$ is called its *conjugate*. Obviously, if $a - b$ is the conjugate of $a + b$, it follows that $a + b$ is the conjugate of $a - b$.

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Since $(a + b) \cdot (a - b) = a^2 - b^2$, multiplying an expression consisting of the sum of two second-order radicals by its yields an expression free of radicals. Thus, suppose a = $\sqrt{p}$ and b = $\sqrt{q}$. Then $(a + b) \cdot (a - b) = (\sqrt{p} + \sqrt{q}) \cdot (\sqrt{p} - \sqrt{q}) = (\sqrt{p})^2 - (\sqrt{q})^2$.

Example 3: $(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) = 3 - 2 = 1$.

Given a fraction where the denominator consists of the algebraic sum of two second-degree radicals, the denominator may be "rationalized" (that is, freed of radicals) by multiplying both numerator and denominator of the fraction by the conjugate of the denominator.

Example 4: Rationalize the denominator of $(\sqrt{5} + \sqrt{3})/(\sqrt{5} - \sqrt{3})$.

SOLUTION:

$$\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{5 + 2\sqrt{5}\cdot\sqrt{3} + 3}{5 - 3} = \frac{8 + 2\sqrt{15}}{2} = 4 + \sqrt{15}$$

The technique works equally well when only one second-degree radical is present in the denominator.

Example 5: Rationalize the denominator of $(\sqrt{7}) + 2\sqrt{3}) / (2 - \sqrt{5})$.

SOLUTION:

$$\frac{\sqrt{7}+2\sqrt{3}}{2-\sqrt{5}} = \frac{\sqrt{7}+2\sqrt{3}}{2-\sqrt{5}} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}$$

$$= \frac{2\sqrt{7} + \sqrt{7}\cdot\sqrt{5} + 4\sqrt{3} + 2\sqrt{3}\cdot\sqrt{5}}{4 - 5}$$

$$= \frac{2\sqrt{7} + \sqrt{35} + 4\sqrt{3} + 2\sqrt{15}}{-1}$$

$$= -4\sqrt{3} \; - 2\sqrt{7} \; - 2\sqrt{15} \; - 35$$

where the radicals have been arranged in increasing value.

The square of an expression consisting of the sum of two second-degree radicals will contain only one radical. Thus,

$$(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + 2\sqrt{a}\cdot\sqrt{b} + (\sqrt{b})^2$$

$$= a + b + 2\sqrt{ab}$$

This fact may be made use of to rationalize a denominator consisting of the sum of three second-degree radicals. The work is performed in two steps.

Example 6: Rationalize the denominator of $\sqrt{2}/(\sqrt{3}+\sqrt{5}-\sqrt{7})$.
SOLUTION:

$$\frac{\sqrt{2}}{\sqrt{3}+\sqrt{5}-\sqrt{7}} = \frac{\sqrt{2}}{(\sqrt{3}+\sqrt{5})-\sqrt{7}}$$

$$= \frac{\sqrt{2}}{(\sqrt{3}+\sqrt{5})-\sqrt{7}} \cdot \frac{(\sqrt{3}+\sqrt{5})+\sqrt{7}}{(\sqrt{3}+\sqrt{5})+\sqrt{7}}$$

$$= \frac{\sqrt{2}\cdot[(\sqrt{3}+\sqrt{5})+\sqrt{7}]}{(\sqrt{3}+\sqrt{5})^2-(\sqrt{7})^2}$$

$$= \frac{\sqrt{6}+\sqrt{10}+\sqrt{14}}{3+2\sqrt{15}+5-7} = \frac{\sqrt{6}+\sqrt{10}+\sqrt{14}}{1+2\sqrt{15}}$$

We now rationalize the denominator of this new fraction:

$$\frac{\sqrt{6}+\sqrt{10}+\sqrt{14}}{1+2\sqrt{15}} = \frac{\sqrt{6}+\sqrt{10}+\sqrt{14}}{1+2\sqrt{15}} \cdot \frac{1-2\sqrt{15}}{1-2\sqrt{15}}$$

$$= \frac{\sqrt{6}+\sqrt{10}+\sqrt{14}-2\sqrt{90}-2\sqrt{150}-2\sqrt{210}}{1-5\cdot 15}$$

$$= \frac{\sqrt{6}+\sqrt{10}+\sqrt{14}-6\sqrt{10}-10\sqrt{6}-2\sqrt{210}}{-59}$$

$$= \frac{9\sqrt{6}+5\sqrt{10}-\sqrt{14}+2\sqrt{210}}{59}$$

This technique may be extended to cover denominators consisting of the sum of more than three second-degree radicals, but the work becomes quite complicated.

7.13 DIVISION OF COMPLEX NUMBERS

The concept of the conjugate of an expression applies also to complex numbers. Thus. given any complex number $a + bi$, the complex number $a - bi$ is called its conjugate. It is clear that the conjugate of $a - bi$ is $a + bi$. Real numbers are their own conjugates: Since any real number may be written as $a + 0 \cdot i = a$,[1] its conjugate would be $a - 0 \cdot i = a$, and hence any real number is its own conjugate.

The product of a complex number and its conjugate is a real number.

$$(a+bi)\cdot(a-bi) = a^2 - abi + abi - (bi)^2$$

$$= a^2 - b^2i^2$$

$$= a^2 - b^2(-1) = a^2 + b^2$$

[1] $a \cdot$ a cow $= a$. This kind of reasoning is totally superfluous, not to mention invalid.

This result shows us that the sum of two squares can be factored, but only at the price of introducing imaginary coefficients in the factors. In general, this is not desirable, but may be quite useful on occasion. Thus,

$$16x^2 + 9y^4 = (4x + 3y^2i) - (4x - 3y^2i)$$

The use of conjugates enables us to find the quotient of two complex numbers, by a process analogous to that of rationalizing the denominator of a fraction when this denominator contains second-degree radicals.

Example 1: Divide 2 - 3*i* by 1 + 2*i*.

SOLUTION:

$$(2 - 3i) \div (1 + 2i) = \frac{1 - 3i}{1 + 2i}$$

$$= \frac{2 - 3i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{2 - 4i - 3i - 6}{1^2 + 2^2}$$

$$= \frac{-4 - 7i}{5} = \frac{4}{5} - \frac{7}{5}i$$

EXERCISE 7.6

Perform the indicated multiplications, and express your answer in simplest form.

1. $\sqrt{2} \cdot (\sqrt{3} + \sqrt{5})$
2. $\sqrt{7} \cdot (\sqrt{5} - 2\sqrt{14})$
3. $(1 + \sqrt{8}) \cdot (2 - \sqrt{12})$
4. $(3 - 2\sqrt{5}) \cdot (4 - \sqrt{5})$
5. $(2 + 3\sqrt{3}) \cdot (\sqrt{5} - \sqrt{3})$
6. $(2\sqrt{5} + \sqrt{7}) \cdot (\sqrt{10} - 2)$
7. $(\sqrt{8} - 2\sqrt{5}) \cdot (\sqrt{2} + \sqrt{10})$
8. $(\sqrt{3} + 2\sqrt{7}) \cdot (\sqrt{6} + \sqrt{7})$
9. $(2 - \sqrt{5} + \sqrt{8}) \cdot (\sqrt{3} - \sqrt{10} + \sqrt{12})$
10. $(\sqrt{3} - \sqrt{7} + 2\sqrt{8}) \cdot (\sqrt{2} + \sqrt{6} - \sqrt{10})$

Rationalize the denominators of the given factions. Reduce all radicals in the numerator to simplest form.

11. $\dfrac{\sqrt{3} + 2\sqrt{5}}{\sqrt{7}}$
12. $\dfrac{2\sqrt{8} - \sqrt{5}}{\sqrt{2}}$

17. $\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

18. $\dfrac{\sqrt{2}+\sqrt{7}}{\sqrt{5}-\sqrt{3}}$

19. $\dfrac{1+\sqrt{2}}{\sqrt{3}+\sqrt{5}-\sqrt{6}}$

20. $\dfrac{4-\sqrt{3}}{\sqrt{2}-\sqrt{5}+\sqrt{7}}$

Perform the following division by complex numbers.

21. $(2) \div (1 + i)$

22. $(5) \div (2 - i)$

23. $(i) \div (1 - i)$

24. $(- 7i) \div (3 + 2i)$

25. $(1 + i) \div (1 + 2i)$

26. $(1 - i) \div (2 - i)$

27. $(2 - 3i) \div (3 - 2i)$

28. $(5 + 2i) \div (7 - 2i)$

29. $(3 + 4i) \div (1 - 3i)$

30. $(2 + 7i) \div (4 - i)$

7.14 SCIENTIFIC NOTATION

An important application of exponents is made in a system of expressing numbers which is known as scientific notation. In this method of representation, any number is written as the product of a number between one and ten, multiplied by some integer power of 10. The following examples illustrate Be method

Decimal Notation	*Scientific Notation*
2472,851,000	2.472851×10^9
35,000	3.5×10^4
16.42	1.642×10
2.51	2.51
0.71	7.1×10^{-1}
0.00052	5.2×10^{-4}
0.00000081	8.1×10^{-7}

Scientific notation has numerous advantages in scientific and engineering work. In the first place, when very large or very small numbers are to be written, scientific notation is usually much more compact than ordinary decimal representation, and also much easier to read. Thus for example, the number 47,200,000,000,000,000 is both space-consuming and difficult to gasp accurately, whereas 4.72×10^{15} is much less unwieldy. Similarly, the number 0.000,000,000,000,000,0015, written in scientific notation, becomes 1.5×10^{-18}, with a substantial gain in both compactness and clarity.

Scientific notation is of great assistance in dealing with logarithms, as will be seen later. It is of substantial aid in performing computations involving very large and/or very small numbers.

Scientific notation is based on the following table:

Powers of 10	Decimal Notation	Intermediate Value
10^6	1,000,000	5,500,000 = 5,5 x 10^6
10^5	100,000	550,000 = 5.5 x 10^5
10^4	10,000	55,000 = 5.5 x 10^4
10^3	1,000	5,500 = 5.5 x 10^3
10^2	100	550 = 5.5 x 10^2
10^1	10	55 = 5.5 x 10^1
10^0	1	5.5 = 5.5 x 10^0
10^{-1}	0.1	0.55 = 5.5 x 10 - 1
10^{-2}	0.01	0.055 = 5.5 x 10 - 2
10^{-3}	0.001	0.005,5 = 5.5 x 10 - 3
10^{-4}	0.000,1	0.000,55 = 5.5 x 10 - 4
10^{-5}	0.000,01	0.000,055 = 5.5 x 10 - 5

From this table we easily verify the following rule:

To write a given number in a scientific notation, move the decimal point to the right of the left-most non zero digit of the given number. Count the digits between the original decimal point and the new decimal point; this is the power of 10 required. If the new decimal point is to the left of the original decimal point, this power is positive; if the new decimal point is to the right of the original decimal point, this power is negative.

Usually, in expressing a number in scientific notation, 10^1 is written simply as 10, and 10^0 is omitted entirely.

To rewrite a number in ordinary decimal notation, when we are given its scientific notation representation, we proceed as follows, as we verify easily from the foregoing table:

To change a given number from scientific notation to ordinary decimal notation, move the decimal point as many digits to the right as the power of ten, if this is positive, or to the left as the power of 10, if this is negative. Supply zeros to make up the required number of places, if necessary.

The following examples illustrate the method and its application.

Example 1: Change 3,412.016 to scientific notation.

3.412.016

↑ ↑

new original
decimal decimal
point point

There are three digits between, the original decimal point and the new decimal point, and the new decimal point is to the left of the original point. The required power of 10 is, therefore, +3.

$\therefore$ 3,412.016 = 3.412016 x 10^3

Example 2: Rewrite 0.000123 in scientific notation.

SOLUTION:

0.0001.23

↑ ↑

original new

decimal decimal

point point

Four digits lie between the original and the new decimal points, and the new decimal point is to the right of the original decimal point.

$\therefore$ 0.000123 = 1.23 x10^{-4}

Example 3: Rewrite 3.16 x 10^5 in decimal notation.

SOLUTION:

5 digits to the right, supplying 3 zeros

3.16000.

↑ ↑

original new

decimal decimal

point point

$\therefore$3.16 x 10^5 = 316,000

Example 4: Rewrite 7.326 x 10^{-2} in decimal notation.

SOLUTION: two digits to the left, supplying 1 zero

$\mathbf{0.\overbrace{07}.326}$

↑ ↑

new original

decimal decimal

point point

Example 5: Compute (37.1 x 0.0562)/10,568, using scientific notation. Give your answer in both ordinary and scientific notation.

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SOLUTION: In scientific notation,

$$\frac{37.1 \times 0.0562}{10{,}568} = \frac{3.71 \times 10 \times 5.62 \times 10^{-2}}{1.0568 \times 10^{4}}$$

$$= \frac{3.71 \times 562}{1.0568} \ \frac{10 \times 10^{2}}{10^{4}} \quad = 19.73 \times 10^{-5}$$

$= 1.973 \times 10^{-4}$, in scientific notation.

In ordinary notation,

$$1.973 \times 10^{-4} = 0.0001973$$

EXERCISE 7.7

Write the following numbers in scientific notation.

1. 741.83
2. 0.00216
3. 58,431.7
4. 985,700,000
5. 0.000007162
6. 143,210,000,000,000
7. 0.000,000,003,126
8. 4.173
9. 0.112
10. 0.0022731

Rewrite the following in ordinary decimal notation.

11. 2.641×10^{3}
12. 1.5556×10^{-7}
13. 8.92×10^{8}
14. 4.6127×10^{-5}
15. 9.99912×10^{4}
16. 6.7×10^{-2}
17. 8.001×10^{4}
18. 1.12×10
19. 4.07
20. 3.415×10^{2}